Option Prices and Costly Short-Selling*

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Abstract

Much empirical evidence shows that short-selling costs have significant effects on option prices, though theory is scant. Towards this, we provide an analysis of option prices with costly short-selling and option marketmakers. In our model, as in practice, short-sellers incur a shorting fee to borrow stock shares from investors who do not necessarily lend all their long positions (partial lending). Our model delivers simple, closed-form, unique option bid and ask prices that represent marketmakers' expected cost of hedging, and are in terms of and preserve the well-known properties of the Black-Scholes prices. Consistently with empirical evidence, we show that bid-ask spreads of typical options and apparent put-call parity violations are increasing in the shorting fee. We also find that option bid-ask spreads are decreasing in the partial lending, and the effects of costly short-selling on option bid-ask spreads are more pronounced for relatively illiquid options with lower trading activity. We then apply our model to the recent 2008 short-selling ban period and obtain implications consistent with the documented behavior of option prices of banned stocks. Finally, our quantitative analysis reveals that the effects of costly short-selling on option prices are economically significant for expensive-to-short stocks and also sheds light on the behavior of option prices and apparent mispricings of the Palm stock in 2000.

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1 Introduction

Short-selling activity has much grown over the last several decades and now accounts for a significant fraction of trades.[1] A pervasive imperfection in selling a stock short is that it is costly (discussed below), and growing empirical evidence shows that these costs have significant effects on security prices, particularly on the prices of exchange-traded options. The evidence includes option bid-ask spreads and put option implied volatilities being increasing in the short-selling costs (Evans, Geczy, Musto, and Reed (2007), Lin and Lu (2016)), and apparent put-call parity violations being increasing in the short-selling costs (Lamont and Thaler (2003), Ofek, Richardson, and Whitelaw (2004), Evans, Geczy, Musto, and Reed (2007)). Similar evidence during the 2008 short-selling ban period shows that option bid-ask spreads and apparent put-call parity violations of banned stocks were higher than those of unbanned stocks (Battalio and Schultz (2011), Grundy, Lim, and Verwijmeren (2012), Lin and Lu (2016)). During this period, marketmakers also asymmetrically adjusted the option prices of banned stocks by decreasing their call bid prices but increasing their put ask prices (Battalio and Schultz (2011)). On the theory side, however, there is no existing work to reconcile these findings, nor any work, in that regard, that provides option prices incorporating short-selling costs in a straightforward manner. Towards this, in this paper, we provide an analysis of option prices in the presence of costly short-selling. The model we develop leads to tractable, intuitive, closed-form option prices, and importantly, delivers implications that support all the empirical evidence discussed above.

Specifically, we adopt the classic Black-Scholes option pricing framework (Black and Scholes (1973)) and incorporate costly short-selling in the underlying stock, following standard short-selling and stock lending market practices. Short-sellers incur a *shorting fee* to borrow shares from investors who are long in the stock. Those investors who are long in the stock do not necessarily lend all their shares but only a part of them. This *partial lending* is a key feature of our analysis and follows from the fact that most stocks in reality have excess

[1]For instance, Hanson and Sunderam (2014) show that the average short interest ratios for NYSE and AMEX stocks have more than quadrupled from 1988 to 2011, and Diether, Lee, and Werner (2009) report that roughly 30% of the trading volume in NYSE and NASDAQ was due to short-selling in 2005. Relatedly, Saffi and Sigurdsson (2010) report that the amount of global supply of lendable shares in December 2008 was $15 trillion (about 20% of the total market capitalization) and $3 trillion of this amount was lent out to short-sellers.

supply of lendable shares (D'Avolio (2002), Saffi and Sigurdsson (2010)). An investor then effectively pays a different rate for short-selling a share as compared to the rate she earns from holding a share long. We demonstrate that this difference plays an important role in option prices and in the ability of our model in supporting the empirical evidence.

We first demonstrate that with costly short-selling, standard no-arbitrage restrictions alone cannot determine call and put option prices but only lead to lower and upper bounds for them. Notably, we identify these bounds in terms of Black-Scholes prices. The lower bounds are the proceeds from the classic hedge portfolio (in the underlying stock and bond perfectly hedging the options at their maturities) for option buyers, and the upper bounds the costs of the hedge portfolio for option sellers. The no-arbitrage ranges for option prices arise because the hedge portfolio costs for option sellers and proceeds for option buyers differ – one is long while the other is short in the stock. We further show that with costly short-selling, no-arbitrage call prices are lower and put prices higher than their Black-Scholes model counterparts. The reason is that call prices are at most the cost of the option sellers' hedge portfolios and put prices are at least the proceeds from the option buyers' hedge portfolios. Since both portfolios are long in the stock for which there is a benefit, the call option sellers' hedge portfolio costs are lower and the put option buyers' hedge portfolio proceeds higher as compared to those in the Black-Scholes economy.

To obtain unique prices, we impose more structure and introduce option marketmakers in our framework, following the related literature and the actual marketmaking in exchange-traded option markets. Marketmakers are competitive and continuously quote bid and ask option prices that result in zero expected profit from each possible sell or buy order. To hedge the risk in each order, marketmakers form a hedge portfolio, which is either held until the option maturity or liquidated prior to that when a subsequent offsetting order arrives. Hence, each sell or buy order is perfectly hedged at its maturity in two ways, either via a hedge portfolio or via a subsequent offsetting order. We first obtain an intuitive representation for option prices with bid and ask prices being the marketmakers' expected cost of hedging sell and buy orders, respectively. This is a notable generalization of the Black-Scholes option prices, which are equal to the cost of their hedge portfolios.

We then obtain unique, closed-form, option bid and ask prices, and show that they have simple forms in terms of the Black-Scholes prices. Consequently, these option prices preserve the well-known, useful properties of the Black-Scholes model, including prices not depending on investor preferences or the underlying stock mean returns, the signs of option deltas, gammas, vegas, and rhos all being as in the Black-Scholes model. We also show that marketmakers quote lower bid-ask spreads than their no-arbitrage ranges by setting higher bid and lower ask option prices than the respective costs of the hedge portfolios. This implies that investors have incentives to trade with marketmakers rather than to replicate the options themselves within our model. This is in contrast to the Black-Scholes model in which options do not offer any cost advantages over and above their replicating alternatives. Competitive marketmakers are able to offer these more favorable prices to investors because it is less costly for them to perfectly hedge their trades through offsetting orders as compared to hedge portfolios. Furthermore, our closed-form option prices enable us to explore the implications of costly short-selling in a tractable way and relate them to the documented empirical evidence, as discussed below.

Looking more closely at the behavior of the unique option prices, we obtain several noteworthy implications. We find that both the call and put bid-ask spreads are increasing in the shorting fee for typical options, consistent with empirical evidence (Evans, Geczy, Musto, and Reed (2007), Lin and Lu (2016)). This is because, an increase in the shorting fee not only increases short-selling costs but also partially increases the benefit of holding a share long. Hence, the marginal effect of the shorting fee on hedge portfolios that require short-selling the stock is greater, leading to higher bid-ask spreads for typical options. We further show that put bid and ask prices, and hence the put option implied volatilities, are increasing in the shorting fee since a higher shorting fee increases both the cost of and the proceeds from the hedge portfolios for a put seller and buyer, respectively, in line with the empirical evidence (Evans, Geczy, Musto, and Reed (2007), Lin and Lu (2016)). We also show that implied stock prices decrease in the shorting fee, and hence deviate more from the underlying stock prices which then lead to higher apparent put-call parity violations, as also empirically documented (Lamont and Thaler (2003), Ofek, Richardson, and Whitelaw (2004), Evans, Geczy, Musto, and Reed (2007)). We also provide a novel testable implication that the call and put bid-ask spreads are decreasing in the partial lending. The difference from the shorting fee implication arises because an increase in the partial lending only increases the benefit of holding a share long but has no effect on hedge portfolios that require short-selling the stock. Finally, we demonstrate that the effects of short-selling costs on option bid-ask spreads are more pronounced for relatively illiquid options with lower trading activity. This occurs because marketmakers are more likely to hedge the relatively illiquid options via hedge portfolios, through which short-selling costs affect option prices directly.

We then apply our model to the widely-studied 2008 US short-selling ban period, during which option marketmakers were still allowed to short-sell. The evidence indicates that the short-selling ban reduced (roughly halved) the short-selling activity while increasing (roughly doubling) the shorting fee of banned stocks (Boehmer, Jones, and Zhang (2013), Harris, Namvar, and Phillips (2013), Kolasinski, Reed, and Thornock (2013)). Given that, we first show that both the call and put bid-ask spreads and apparent put-call parity violations of banned stocks are higher than those of unbanned stocks, consistent with empirical evidence (Battalio and Schultz (2011), Grundy, Lim, and Verwijmeren (2012), Lin and Lu (2016)). We then demonstrate an asymmetric effect of the ban on the option prices of the banned stocks by showing that the call bid prices decrease more than the ask prices, while the put ask prices increase more than the bid prices, also consistent with empirical evidence (Battalio and Schultz (2011)). These results arise because the short-selling ban only affects those hedge portfolios that are short, but does not affect the ones that are long in the stock as they earn the same rate per share. This reduces the proceeds from the hedge portfolio for marketmakers when they buy a call and increases the costs of the hedge portfolio when they sell a put option, leading to a relatively higher decrease in the call bid while a relatively higher increase in the put ask prices. This mechanism also leads to higher option bid-ask spreads and higher apparent put-call parity violations for banned stocks.

Finally, we quantify our model and demonstrate that the effects of short-selling costs on option prices are economically significant for expensive-to-short stocks (stocks in the highest shorting fee decile). We also apply our model to the well-publicized event of extreme short-selling in the Palm stock in 2000, during which there were apparent violations of the law of one price (Lamont and Thaler (2003)). We demonstrate that roughly half of the observed price deviations of the Palm stock could be due to the costly short-selling, implying that the combined effects of all the other risks, costs and considerations could amount to the remaining half.

There is a large theoretical literature investigating the effects of various market imperfections on option prices. These include looking at the effects of taxes (Scholes (1976)), transaction costs (Leland (1985), Hodges and Neuberger (1989), Boyle and Vorst (1992), Bensaid, Lesne, Pagès, and Scheinkman (1992), Edirsinghe, Naik, and Uppal (1993), Davis, Panas, and Zariphopoulou (1993), Soner, Shreve, and Cvitanić (1995), Constantinides and Perrakis (2002)), trading constraints including short-selling restrictions (Karatzas and Kou (1996), Broadie, Cvitanić, and Soner (1998)), different interest rates for borrowing and lending (Bergman (1995)), funding, collateral and margin requirements (Piterbarg (2010), Bielecki and Rutkowski (2015), Leippold and Su (2015)). Effective market incompleteness implied by these imperfections in general leads to no-arbitrage ranges rather than unique option prices. This is typically addressed, if at all, by introducing a utility maximization problem which often times leads to complex option prices that depend on investor preferences. In contrast, the markets are complete in our framework since it is still possible to perfectly hedge the option payoffs by trading in the underlying stock and the bond. Hence, standard no-arbitrate restrictions along with a perfectly-hedging marketmaking function suffice to obtain unique closed-form preference-free option bid and ask prices.

More closely related works that study the effects of frictions in the short-selling markets on option prices are Avellaneda and Lipkin (2009) and Jensen and Pedersen (2016). Avellaneda and Lipkin study how option prices are affected by the "buy-in" risk that short-sellers may have to close their short positions prematurely, but do not consider the effects of the shorting fee and partial lending as we do. On the other hand, Jensen and Pedersen overturns the classic result of Merton (1973) by showing that in the presence of shorting fees, as well as margin and funding costs, it may in fact be optimal to exercise an American call option early. However, differently from our work, their focus is on the optimality of the early exercise of American options rather than studying the effects of the shorting fee and partial lending on European options. Moreover, differently from these works, we consider the marketmaking facility to obtain bid and ask prices for options and relate them to the empirical evidence on short-selling costs.

The remainder of the article is organized as follows. Section 2 presents our model with costly short-selling and Section 3 introduces option marketmakers and provides the unique bid and ask prices. Section 4 investigates the behavior of the option prices, and Section 5 the 2008 short-selling ban and the quantitative analysis. Section 6 concludes. Appendix A contains the proofs and Appendix B provides additional quantitative analysis.

2 No-Arbitrage Option Prices with Costly Short-Selling

In this Section, we adopt the classic Black-Scholes framework and incorporate costly short-selling in the underlying stock. Short-sellers incur a shorting fee to borrow shares from investors who are long in the stock. Those investors who are long in the stock do not necessarily lend all their shares but only a part of them. We demonstrate that with costly short-selling, option prices that admit no-arbitrage fall within a range, for which we identify its bounds to be in terms of Black-Scholes prices.

2.1 Economy with Costly Short-Selling

In the classic Black-Scholes economy, the securities market includes a riskless bond and a (non-dividend paying) stock whose price processes B and S follow

$$dB_t = B_t r dt, \tag{1}$$

$$dS_t = S_t \left[\mu dt + \sigma d\omega_t\right], \tag{2}$$

where r is the constant riskless interest rate, μ and σ are the constant mean return and the return volatility of the stock, respectively, and ω is a standard Brownian motion. Trading in these securities is unrestricted. That is, there are no taxes, transaction costs, restrictions on borrowing, and in particular short-selling the stock is costless.

We incorporate short-selling costs into this economy by following standard short-selling and stock lending market practices. Short-sellers borrow shares from investors who are long in the stock. All short-selling proceeds are kept as collateral in an account that earns the riskless interest rate r. This interest income is shared between the lender and the short-seller. The lender's account earns the *shorting fee* rate $\phi > 0$, and the short-seller's account earns the rebate rate $r - \phi$.[2] On the other hand, investors who are long in the stock do not necessarily lend all their shares but only a fraction $0 \leq \alpha < 1$ of them, where henceforth,

[2]Note that the rebate rate can be negative and the rate short-sellers are effectively paying to lenders is the shorting fee ϕ as it is the foregone interest rate for them. The exact mechanics of stock short-selling are somewhat more involved but its essentials are captured by our formulation above (see Reed (2013) for an extensive discussion of short-selling). We discuss how to adjust our model for additional considerations in these markets in Remark 1.

we refer to α as the *partial lending* parameter. The partial lending feature follows from the fact that most stocks in reality have excess supply of lendable shares (D'Avolio (2002), Saffi and Sigurdsson (2010)). Hence, even if investors attempt to do so, they may not be able to successfully lend all their long stock positions.[3] In sum, an investor effectively pays a rate ϕ for short-selling a share but only earns the rate $\alpha\phi$ from holding a stock share long in our economy with costly short-selling.[4] We demonstrate that the difference between the cost of short-selling and the benefit of holding a share long plays an important role in option prices and in the ability of our model in supporting the empirical evidence.

2.2 No-Arbitrage Option Prices

We consider standard European-style call and put options written on the stock with a strike price K and a maturity date T. For the call the buyer's payoff is $\max\{S_T - K, 0\}$ and the seller's $-\max\{S_T - K, 0\}$, while for the put the buyer's payoff is $\max\{K - S_T, 0\}$ and the seller's $-\max\{K - S_T, 0\}$ at the maturity date. An option price is said to admit arbitrage if the investors can form a self-financing portfolio to obtain a strictly positive initial profit with zero payoff at the option maturity by trading at that option price.[5] In the classic Black-Scholes economy without costly short-selling, the no-arbitrage restriction alone is sufficient to uniquely determine the option prices, given by the cost of the hedge portfolio, a self-financing portfolio in the underlying stock and the riskless bond which perfectly hedges (offsets) the option seller's payoff at the maturity date. Proposition 1 reports the no-arbitrage option prices in our economy.

[3]The partial lending feature can also be justified due to the standard equilibrium condition for security markets, that is, since short-sellers need to sell the shares back to other long holders, not every long position can be lent to short-sellers in equilibrium. As for the size of the excess supply, Saffi and Sigurdsson (2010) report that the amount of global supply of lendable shares in December 2008 was \$15 trillion (about 20% of the total market capitalization) and only \$3 trillion of this amount was actually lent out. Saffi and Sigurdsson also report that the average fraction of outstanding shares lent out in their sample was 8.91% for the US and 5.75% for the world, which could proxy our partial lending parameter for a typical stock.

[4]Note that the short-seller pays the shorting fee to the lender over time until the short position is closed. This is in contrast to the commonly considered financial friction of (proportional) transaction costs in which the cost is incurred only when a trade takes place.

[5]This arbitrage definition is sufficient for our purposes and follows from its more formal definition in standard textbooks (e.g., Duffie (2001)).

Proposition 1 (No-arbitrage option prices). *In the economy with costly short-selling, no-arbitrage call and put prices, C_t and P_t, satisfy*

$$C_t^{BS}(\phi) \leq \; C_t \; \leq C_t^{BS}(\alpha\phi), \tag{3}$$
$$P_t^{BS}(\alpha\phi) \leq \; P_t \; \leq P_t^{BS}(\phi), \tag{4}$$

where $C_t^{BS}(q)$ and $P_t^{BS}(q)$ denote the standard Black-Scholes call and put prices adjusted for the constant dividend yield q, respectively, and are given by

$$C_t^{BS}(q) \;=\; S_t e^{-q(T-t)}\Phi(d_1(q)) - Ke^{-r(T-t)}\Phi(d_2(q)), \tag{5}$$
$$P_t^{BS}(q) \;=\; -S_t e^{-q(T-t)}\Phi(-d_1(q)) + Ke^{-r(T-t)}\Phi(-d_2(q)), \tag{6}$$

where $\Phi(.)$ is the standard normal cumulative distribution function and

$$d_1(q) = \frac{\ln(S_t/K) + \left(r - q + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \qquad \text{and} \qquad d_2(q) = d_1(q) - \sigma\sqrt{T-t}. \tag{7}$$

In the special case of full lending ($\alpha = 1$), no-arbitrage call and put prices are unique and given by

$$C_t = C_t^{BS}(\phi), \qquad P_t = P_t^{BS}(\phi). \tag{8}$$

Consequently, in the economy with costly short-selling,

i) The call price upper bound is decreasing in both the shorting fee ϕ and partial lending α, while its lower bound is decreasing in the shorting fee ϕ but is not dependent on the partial lending α.

ii) The put price upper bound is increasing in the shorting fee ϕ but is not dependent on the partial lending α, while its lower bound is increasing in both the shorting fee ϕ and partial lending α.

iii) The call prices are lower, while the put prices are higher than the corresponding prices in the Black-Scholes model without costly short-selling.

Proposition 1 shows that with costly short-selling, the no-arbitrage restriction alone is not sufficient to determine option prices uniquely. Call and put prices that admit no-arbitrage now fall within a range whose bounds are identified in terms of Black-Scholes prices. The upper bounds in (3)–(4) are the costs of the hedge portfolio (the self-financing portfolio in the underlying stock and the riskless bond which perfectly hedges the option payoff at the maturity date) for option sellers. If options are traded at prices higher than these upper bounds then they admit arbitrage, because selling the option and forming the hedge portfolio at a lower cost leads to a positive initial profit with no payoff at the option maturity date. The lower bounds in (3)–(4) are the proceeds from the hedge portfolio for option buyers and differ from the upper bounds. If options are traded at prices lower than these lower bounds then they admit arbitrage, since buying the option and receiving a higher amount by forming the hedge portfolio leads to a positive initial profit with no payoff at the option maturity date. On the other hand, if options are traded at prices within these ranges then they admit no-arbitrage, since selling or buying the option and forming the hedge portfolio can lead to a zero initial profit at most.[6]

With costly short-selling, these ranges for no-arbitrage option prices arise because the cost of short-selling (ϕ) and the benefit of holding a share long ($\alpha\phi$) are not the same. Therefore, the cost of the hedge portfolio for option sellers and the proceeds from the hedge portfolio for option buyers are different, as when one hedge portfolio is long while the other is short in the underlying stock. This is in contrast to the classic Black-Scholes economy without costly short-selling, as well as the special case of full lending in our economy ($\alpha = 1$) as (8) illustrates, for which no-arbitrage option prices are unique. Uniqueness is due to the fact that the cost of short-selling and the benefit of holding a share long are the same and equal to either zero (Black-Scholes economy) or to the shorting fee (full lending in our economy).

Turning to the roles of the shorting fee ϕ and partial lending α, we see that the call price upper bound $C_t^{BS}(\alpha\phi)$ is decreasing in both the shorting fee and partial lending (property (i)). As discussed above, this upper bound is the cost of the call option seller's hedge portfolio that is long in the underlying stock. Therefore, an increase in either the shorting fee or partial lending increases the benefit of holding a share long, which reduces the cost of

[6]As we demonstrate in Table 4 of Appendix B, these no-arbitrage bounds may significantly differ from the Black-Scholes prices. For instance, for a 3-month, at-the-money option written on the typical stock in the highest shorting fee decile of Drechsler and Drechsler (2016), the lower bound for the call option is 11.22% lower than and the upper bound for the put option is 10.46% higher than the corresponding call and put prices in the Black-Scholes model.

the hedge portfolio. On the other hand, the call price lower bound $C_t^{BS}(\phi)$ is the proceeds from the call option buyer's hedge portfolio that is short in the underlying stock. Therefore, an increase in the shorting fee reduces the proceeds. Moreover, as the hedge portfolio is short in the underlying stock, partial lending does not affect it (property (i)). Similar arguments also lead to the put price upper bound $P_t^{BS}(\phi)$ being increasing in the shorting fee but not being affected by partial lending, and its lower bound $P_t^{BS}(\alpha\phi)$ being increasing in both the shorting fee and partial lending (property (ii)).

A notable implication here is that the call price is lower and the put price higher than those in the Black-Scholes model without costly short-selling (property (iii)). The intuition is as follows. A call price is at most the cost of the option seller's hedge portfolio that is long in the stock. Similarly, a put price is at least the proceeds from the option buyer's hedge portfolio that is also long in the stock. Since there is a benefit of holding the stock long, the call option seller's hedge portfolio cost is lower and the put option buyer's hedge portfolio proceeds are higher as compared to those in the Black-Scholes economy without costly short-selling. Hence, with costly short-selling, a call price is at most less than and the put price is at least more than the corresponding prices implied by the Black-Scholes model. Note that, this can also be seen from our earlier results with respect to the shorting fee ϕ in properties (i)–(ii), along with the fact that the Black-Scholes prices arise as a special case of our model with zero shorting fee ($\phi = 0$).

Remark 1 (Additional considerations). To highlight our results as clearly as possible, we did not consider several possible issues, but they can easily be incorporated into our analysis. First, our model can be extended to a setting in which the stock pays a constant dividend yield δ, by adding it to both the shorting fee ϕ and the lending income $\alpha\phi$. For instance, a call price that admits no-arbitrage would satisfy $C_t^{BS}(\phi + \delta) \leq C_t \leq C_t^{BS}(\alpha\phi + \delta)$. Second, in our formulation 100% of the short-selling proceeds are kept as collateral. This rate is very close to the actual practice in the US for domestic stocks, as Reed (2013) reports lenders typically require 102% of the short-selling proceeds as a collateral to help protect themselves. Our model can be generalized to any constant collateral rate κ by simply multiplying the shorting fee ϕ by κ. For instance, in this case a call price that admits no-arbitrage satisfies $C_t^{BS}(\phi\kappa) \leq C_t \leq C_t^{BS}(\alpha\phi\kappa)$. Third, in our model the lender gets all of the shorting fee ϕ upon successfully lending a share. In reality, this is true for some large institutions with internal lending departments which directly lend to short-sellers. Other lenders typically use an agent bank/brokerage and get only a fraction of the shorting fee, with the rest going to the agent bank/brokerage for providing this service. Reed (2013) reports that these lenders typically get 75% of the shorting fee. Incorporating this feature into our model is straightforward by multiplying the partial lending parameter α by a constant fraction γ capturing the lender's share of shorting fee. For instance, in this case a call price that admits no-arbitrage satisfies $C_t^{BS}(\phi) \leq C_t \leq C_t^{BS}(\alpha\gamma\phi)$.

In our analysis, we only consider standard call and put options as most of the empirical evidence on the effects of costly short-selling is on these. Our analysis, however, is equally valid for other European-style derivatives whose payoffs are monotonically either non-decreasing or non-increasing in the underlying stock, such as forward contracts. Moreover, to keep our analysis comparable to the Black-Scholes economy with constant parameters, we take the shorting fee ϕ and partial lending parameter α to be constants. In reality, the levels of shorting fee and partial lending are likely time varying. Introducing time-variation in these parameters may be addressed by the methodologies employed in option pricing with stochastic dividend yields (e.g., Geske (1978), Broadie, Detemple, Ghysels, and Torrés (2000)).

3 Option Bid and Ask Prices with Marketmakers

As the previous Section illustrates, with costly short-selling, standard no-arbitrage restrictions alone cannot determine option prices but only lead to lower and upper bounds for them. To obtain unique prices, one would need to impose more structure on the economy. Towards that, in this Section we introduce option marketmakers and obtain unique bid and ask option prices. We show that option prices have simple forms, in terms of the familiar Black-Scholes prices, and inherit the well-known, useful properties of the Black-Scholes model. This analysis also enables us to explore the implications of costly short-selling in a tractable way and relate them to the documented empirical evidence as shown in Section 4.

We incorporate option marketmakers in our framework following the actual marketmaking for exchange-traded option markets, such as the CBOE, as well as the related literature.[7]

[7]Options traded in the over-the-counter markets would involve other issues such as search costs and bargaining (e.g., Duffie, Gârleanu, and Pedersen (2005, 2007)) that are not the main focus of our analysis.

There are numerous competitive option marketmakers who stand ready to buy and sell options to fulfill investor orders, and hence facilitate trading at any point in time. Marketmakers continuously quote bid and ask option prices that result in zero expected profit for each possible trade.[8] Since fulfilling investor orders may generate arbitrary (and adverse) positions for marketmakers, marketmakers attempt to hedge the risk in each order by forming a hedge portfolio. The hedge portfolio is either held until the option maturity or liquidated prior to that when a subsequent offsetting order arrives (e.g., a current call buy order's offsetting order is a subsequent call sell order) as the latter also perfectly hedges the incoming order at its maturity. Hence, each sell or buy order is perfectly hedged at its maturity in two ways, either via a hedge portfolio or via a subsequent offsetting order. The first way of hedging risk is what makes options marketmaking different from that in other markets, and as discussed in Section 2 this hedging can still be achieved perfectly in our economy with costly short-selling. The second way, matching offsetting orders, is the more familiar one in marketmaking, particularly for equities.[9]

We model the arrival of offsetting (buy or sell) orders in a simple way as in Bollen, Smith, and Whaley (2004), which in turn is based on the classic microstructure model of Garman (1976). At each time t, the arrivals of offsetting trades have mutually independent exponential distributions with (positive) parameters $\lambda_{Cs}, \lambda_{Cb}, \lambda_{Ps}, \lambda_{Pb}$, representing the arrival rates of an offsetting call sell, call buy, put sell, put buy order, respectively.[10] These arrival rates

[8]Setting prices such that each option trade yields zero expected profit follows from the competitive marketmaking assumption and is consistent with the literature on option marketmakers (Easley, O'Hara, and Srinivas (1998), Johnson and So (2012)). We discuss other possible features of option marketmaking within our framework in Remark 2.

[9]As is well-recognized in the literature, option marketmakers have a greater need for hedging their positions as compared to equity marketmakers. This is because they face far greater inventory holding costs due to higher illiquidity, and implicit leverage of options result in higher and stochastic volatilities (e.g., Jameson and Wilhelm (1992), Cho and Engle (1998), Muravyev (2016)). They also face far greater order imbalances as compared to those in underlying stocks (Lakonishok, Lee, Pearson, and Poteshman (2007)), which can be attributed to inventory risk (Muravyev (2016)). In our model inventory risk does not play a role since marketmakers immediately form a hedge portfolio for each order (availability of second way of perfect hedging).

[10]The arrivals of offsetting orders are also independent from the Brownian motion ω in the underlying stock price dynamics (2). The exponential distribution is commonly used to model the time between the occurrence of events and arises as the distribution of the interarrival times of a Poisson process. For an exponential distribution with parameter λ, the expected arrival time is given by $1/\lambda$. The exponential distribution has the useful "memoryless property", implying that the distribution of the arrival time is independent from the waiting time that has already occurred. In our setting, this allows us to proceed at each time t without

are inherently related to liquidity as one may argue that the more liquid options, which have higher buying/selling activity, are more likely to have higher offsetting order arrival rates (or, equivalently, lower expected arrival times for offsetting orders). We explore the effects of buying/selling activity in Section 4 (Proposition 5). The respective distribution functions of offsetting orders are denoted by $F_{Cs}, F_{Cb}, F_{Ps}, F_{Pb}$, which along with the objective of marketmakers given above are sufficient to determine the option prices as follows. First, the marketmakers compute the current value of profits from each case depending on whether an offsetting order arrives by the option maturity or not. Then, they find the expected profits by taking a weighted average of these profits, where the weights are the probabilities of each case occurring and are characterized by the distribution functions $F_{Cs}, F_{Cb}, F_{Ps}, F_{Pb}$. For instance, while determining the call ask price at time t, the probability of no offsetting call sell order arriving by time T is $1-\int_t^T dF_{Cs}(u)$, and used as the weight for that case. Finally, they set the option bid and ask prices so that the expected profit is zero. Determining the bid and ask prices this way results in the expected cost of perfect hedging representations for option prices as reported in Lemma 1.

Lemma 1 (**Expected cost of hedging representation**). *In the economy with costly short-selling and marketmakers, the call bid and ask prices satisfy*

$$C_t^{Bid} = \int_t^T \left\{ C_t^{BS}(\phi) + V_t \left[C_u^{Ask} - C_u^{BS}(\phi) \right] \right\} dF_{Cb}(u) + C_t^{BS}(\phi) \left(1 - \int_t^T dF_{Cb}(u) \right), \tag{9}$$

$$C_t^{Ask} = \int_t^T \left\{ C_t^{BS}(\alpha\phi) - V_t \left[C_u^{BS}(\alpha\phi) - C_u^{Bid} \right] \right\} dF_{Cs}(u) + C_t^{BS}(\alpha\phi) \left(1 - \int_t^T dF_{Cs}(u) \right). \tag{10}$$

Similarly, the put bid and ask prices satisfy

$$P_t^{Bid} = \int_t^T \left\{ P_t^{BS}(\alpha\phi) + V_t \left[P_u^{Ask} - P_u^{BS}(\alpha\phi) \right] \right\} dF_{Pb}(u) + P_t^{BS}(\alpha\phi) \left(1 - \int_t^T dF_{Pb}(u) \right), \tag{11}$$

$$P_t^{Ask} = \int_t^T \left\{ P_t^{BS}(\phi) - V_t \left[P_u^{BS}(\phi) - P_u^{Bid} \right] \right\} dF_{Ps}(u) + P_t^{BS}(\phi) \left(1 - \int_t^T dF_{Ps}(u) \right), \tag{12}$$

where $V_t[X_u]$ denotes the time-t value of the payoff X_u at time $u \geq t$, and the Black-Scholes call and put prices, $C_t^{BS}(.)$ and $P_t^{BS}(.)$, are as in (5)−(6) *of Proposition 1.*

keeping track of how long one has already waited for the offsetting order.

Lemma 1 indicates that in the economy with costly short-selling and marketmakers, option bid and ask prices are given by the marketmakers' expected cost of hedging sell and buy orders, respectively. This is because the first terms in option prices (9)–(12) are the (weighted) current values of the subsequent offsetting orders (which would perfectly hedge the option in question). The second terms in (9)–(12) are the (weighted) current costs of the hedge portfolios involving the underlying stock (which would also perfectly hedge the option). Summing these two weighted hedging costs, where the weights are given by the distribution functions of offsetting order arrival times, yields the expected cost of hedging representation. In particular, consider the call ask price representation in (10). It is the marketmakers' expected cost of hedging a call option sold at time t (upon the arrival of a buy order at time t), where the expectation is taken with respect to the uncertainty about the offsetting call sell order arrival time given by the distribution function F_{Cs}. The quantity $C_t^{BS}(\alpha\phi) - V_t\left[C_u^{BS}(\alpha\phi) - C_u^{Bid}\right]$ in the first term is the value (cost) of the subsequent offsetting call sell order if it arrives at time $u \geq t$, and the quantity $C_t^{BS}(\alpha\phi)$ in the second term is the cost of the hedge portfolio in the underlying stock. That is, to hedge a call option sold at time t, the marketmaker immediately forms the hedge portfolio at a cost $C_t^{BS}(\alpha\phi)$. If an offsetting call sell order arrives at a subsequent time $u < T$, the marketmaker buys (and keeps) this offsetting option at a price C_u^{Bid} and liquidates the hedge portfolio for a value $C_u^{BS}(\alpha\phi)$. This way the marketmaker hedges via a subsequent offsetting call sell order at a cost $C_t^{BS}(\alpha\phi) - V_t\left[C_u^{BS}(\alpha\phi) - C_u^{Bid}\right]$. If an offsetting call sell order does not arrive by maturity date T, the marketmaker hedges via the hedge portfolio at the cost $C_t^{BS}(\alpha\phi)$.[11] This expected cost of hedging representation is a notable generalization of the standard Black-Scholes model in which option prices are equal to the cost of their hedge portfolios. Even though this representation for option prices is simple and intuitive, to the best of our knowledge it has not been explored previously in the literature with market imperfections.

We note that the lower and upper no-arbitrage bounds for option prices in Proposition 1 (Section 2) arise as the bid and ask prices in the special case when there is no possibility of an offsetting trade. This is intuitive since then the marketmakers can only hedge the options via

[11]Note that the current value operator $V_t[X_u]$ gives the amount required at time-t to form a self-financing portfolio in the underlying stock and the riskless bond to obtain the payoff X_u at time $u \geq t$. Since there is a difference between the cost of short-selling and the benefit of holding a share long, one needs to account for the sign of the payoff X_u while determining its current value as we show in the Appendix.

hedge portfolios, and so they set the option prices equal to the cost of these hedge portfolios. For instance, when there is no possibility of an offsetting call sell order ($\lambda_{Cs} = 0$), the call ask price in (10) coincides with the no-arbitrage upper bound $C_t^{Ask} = C_t^{BS}(\alpha\phi)$ as this is the cost of the hedge portfolio for a call option seller (Section 2). Similarly, when there is no possibility of an offsetting call buy order ($\lambda_{Cb} = 0$), the no-arbitrage lower bound arises as the call bid price $C_t^{Bid} = C_t^{BS}(\phi)$. For the general case with the possibility of offsetting orders, we need to solve the coupled systems (9)–(10) for the call option, and (11)–(12) for the put option, in which the current bid and ask prices depend on the future prices of the other. Solving the above coupled systems involve substituting conjectured (and later verified) bid and ask prices into these systems, differentiating, and solving the resulting systems of two linear first order differential equations simultaneously as shown in the Appendix. This procedure yields the closed-form solutions for the call and put option bid and ask prices, as reported in Proposition 2.

Proposition 2 (**Option bid and ask prices**). *In the economy with costly short-selling and marketmakers, the call bid and ask prices are given by*

$$
\begin{aligned}
C_t^{Bid} &= \left(1 - \mathrm{w}_{t,C^{Bid}}\right) C_t^{BS}(\alpha\phi) + \mathrm{w}_{t,C^{Bid}} C_t^{BS}(\phi), && (13)\\
C_t^{Ask} &= \mathrm{w}_{t,C^{Ask}} C_t^{BS}(\alpha\phi) + \left(1 - \mathrm{w}_{t,C^{Ask}}\right) C_t^{BS}(\phi), && (14)
\end{aligned}
$$

and the put bid and ask prices are given by

$$
\begin{aligned}
P_t^{Bid} &= \left(1 - \mathrm{w}_{t,P^{Bid}}\right) P_t^{BS}(\phi) + \mathrm{w}_{t,P^{Bid}} P_t^{BS}(\alpha\phi), && (15)\\
P_t^{Ask} &= \mathrm{w}_{t,P^{Ask}} P_t^{BS}(\phi) + \left(1 - \mathrm{w}_{t,P^{Ask}}\right) P_t^{BS}(\alpha\phi), && (16)
\end{aligned}
$$

where the Black-Scholes call and put prices, $C_t^{BS}(.)$ and $P_t^{BS}(.)$, are as in (5)–(6) of Proposition 1. The weights for the call bid and ask prices $\mathrm{w}_{t,C^{Bid}}$ and $\mathrm{w}_{t,C^{Ask}}$ are given by

$$
\begin{aligned}
\mathrm{w}_{t,C^{Bid}} &= \frac{\lambda_{Cs}}{\lambda_{Cs} + \lambda_{Cb}} + \frac{\lambda_{Cb}}{\lambda_{Cs} + \lambda_{Cb}} e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)}, && (17)\\
\mathrm{w}_{t,C^{Ask}} &= \frac{\lambda_{Cb}}{\lambda_{Cs} + \lambda_{Cb}} + \frac{\lambda_{Cs}}{\lambda_{Cs} + \lambda_{Cb}} e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)}, && (18)
\end{aligned}
$$

and the weights for the put bid and ask prices $\mathrm{w}_{t,P^{Bid}}$ *and* $\mathrm{w}_{t,P^{Ask}}$ *are given by*

$$\mathrm{w}_{t,P^{Bid}} = \frac{\lambda_{Ps}}{\lambda_{Ps}+\lambda_{Pb}} + \frac{\lambda_{Pb}}{\lambda_{Ps}+\lambda_{Pb}} e^{-(\lambda_{Ps}+\lambda_{Pb})(T-t)}, \tag{19}$$

$$\mathrm{w}_{t,P^{Ask}} = \frac{\lambda_{Pb}}{\lambda_{Ps}+\lambda_{Pb}} + \frac{\lambda_{Ps}}{\lambda_{Ps}+\lambda_{Pb}} e^{-(\lambda_{Ps}+\lambda_{Pb})(T-t)}. \tag{20}$$

Proposition 2 reveals that unique option prices (13)–(16) are weighted-averages of their respective no-arbitrage bounds, which in turn are identified in terms of the Black-Scholes prices (Section 2).[12] These simple representations not only make the option prices easy to compute, but also make them preserve the well-known and widely-employed properties of Black-Scholes prices. In particular, option prices do not depend on investor preferences and the underlying stock mean return μ. Moreover, the signs of the so-called option Greeks, delta, vega, and rho which capture the sensitivities of option prices to the underlying stock price S_t, volatility σ and interest rate r, respectively, as well as the gamma capturing the sensitivity of delta to the underlying stock price, are the same as in the Black-Scholes model.[13] We see that the weights for bid and ask prices (17)–(20) are driven by the arrival rates of both the buy and sell orders, rather than only by the arrival rate of a respective offsetting order. This follows from the fact that the closed-form option prices (13)–(16) are solutions to their respective counterparts (9)–(12) in Lemma 1. There we can see that the current bid and ask prices depend on the future ask and bid prices, respectively, and hence the arrival rates of offsetting sell and buy orders both affect the prices through these weights.

As discussed earlier, in the special case when there is no possibility of an offsetting order, option bid and ask prices become the lower and upper no-arbitrage bounds, respectively. However, when there is the possibility of offsetting orders, option bid and ask prices lie strictly within their no-arbitrage bounds. That is, marketmakers quote lower bid-ask spreads than their no-arbitrage ranges, by setting higher bid and lower ask prices than the respective costs of the hedge portfolios. This is notable as it implies that investors now have incentives to trade with marketmakers rather than to replicate the option payoffs themselves via a portfolio in the underlying stock and the riskless bond, since this way they can sell the same payoff at

[12]As discussed in Section 2, the no-arbitrage upper bounds are the costs of the hedge portfolio for option sellers, and the lower bounds are the proceeds from the hedge portfolio for option buyers.

[13]The sign of the theta, which captures the sensitivity of option prices to time to maturity, is ambiguous as in the Black-Scholes model itself.

a higher price and buy it at a lower price. This is in contrast to the Black-Scholes model in which options do not offer any cost advantages over and above their replicating alternatives constructed with the underlying stock and riskless bond. Competitive marketmakers are able to offer these more favorable prices to investors because it is less costly for them to perfectly hedge their trades through offsetting orders as compared to hedge portfolios. For instance, marketmakers sell the call option at its ask price (14), which is lower than the cost of its hedge portfolio, $C_t^{BS}(\alpha\phi)$. They are willing to do so as there is also the possibility to perfectly hedge the call option sold by buying a call option at a bid price in the future whose current value is less than $C_t^{BS}(\alpha\phi)$, the no-arbitrage upper bound. This reduces the expected cost of perfect hedging a call option sold and leads to a lower call ask price. In addition to offering cost advantages, the possibility of offsetting orders makes the partial lending matter for both the bid and ask option prices. In the special case of there being no possibility of an offsetting order, partial lending does not affect call bid and put ask prices, as the marketmakers' hedge portfolios for buying a call and selling a put are short in the underlying stock. However, with the possibility of offsetting orders, marketmakers also take into account of future offsetting orders whose hedge portfolios are long in the underlying stock, making all prices depend on partial lending.

Remark 2 (Other features of option marketmaking). To obtain our results, we have considered the key features of option marketmaking, and have not incorporated other possible features so as to not unnecessarily confound or complicate our analysis. First, in our model option trades are due to market orders and occur at a fixed size as in Easley, O'Hara, and Srinivas (1998) and Muravyev (2016). Without loss of generality we normalize the trade sizes to one for convenience. Moreover, considering market orders and not additionally the more complex limit orders, which are dependent on prices, turn out to be enough for our analysis and main message. Second, while matching current orders by subsequent offsetting orders, marketmakers do not partially hedge and only classify those orders with the same strike and maturity as offsetting (e.g., a call buy order is matched by a subsequent call sell order with the same strike and maturity). This is because options with different strikes or maturities would not perfectly hedge the current option at its maturity, and hence expose the marketmaker to market risk. This in turn would require additional assumptions on the marketmakers' risk attitude to proceed. Moreover, introducing partial hedging by considering multiple options with different strikes and maturities would significantly complicate the analysis as the Black-Scholes prices are not linear in strike or maturity. Third, to study the effects of costly short-selling in a simple framework that is as close as possible to the standard symmetric information Black-Scholes economy, we do not consider information asymmetry between marketmakers and investors which may also affect the option bid and ask prices as demonstrated in Easley, O'Hara, and Srinivas (1998).

In our model, while determining the current option bid and ask prices, marketmakers always use a subsequent offsetting order to perfectly hedge a current order rather than forming a hedge portfolio for the subsequent order also. This is because, an attempt to form a hedge portfolio for the subsequent offsetting order leads to lower profits, and hence to a suboptimal strategy for them. For instance, as discussed earlier and also demonstrated in the proof of Lemma 1 in the Appendix, while determining the current call ask price, if an offsetting call sell order arrives at a subsequent time $u < T$, the marketmaker buys this offsetting call option at a bid price C_u^{Bid} and liquidates the hedge portfolio for the current option at a value $C_u^{BS}(\alpha\phi)$. However, if the marketmaker were to hedge the offsetting order via a new hedge portfolio by also keeping its existing hedge portfolio until maturity, this strategy would lead to a lower profit. This is because now it is not liquidating the initial hedge portfolio for $C_u^{BS}(\alpha\phi)$ but instead receiving less $C_u^{BS}(\phi)$ to perfectly hedge the call option bought at time u which requires short-selling the underlying stock.

4 Behavior of Option Bid and Ask Prices

In this Section, we investigate the behavior of the option prices obtained in Section 3 in terms of the shorting fee, partial lending and the arrival rates of offsetting orders. Consistent with empirical evidence, we first show that both the call and put bid-ask spreads are increasing in the shorting fee for typical options. We then demonstrate that the stock prices implied by the option prices decrease in the shorting fee, and hence deviate more from the underlying stock prices which then lead to higher apparent put-call parity violations, as also empirically documented. Furthermore, we provide a novel testable implication that call and put bid-ask spreads are decreasing in the partial lending. We also show that the effects of short-selling costs on option bid-ask spreads are more pronounced for relatively illiquid options with lower trading activity.

In this Section, in addition to presenting the effects of costly short-selling on option prices, we also present our results for the (options) implied stock prices using the well-known put-call parity relation, which yields the implied stock bid and ask prices as

$$\tilde{S}_t^{Bid} \equiv C_t^{Bid} - P_t^{Ask} + Ke^{-r(T-t)}, \tag{21}$$

$$\tilde{S}_t^{Ask} \equiv C_t^{Ask} - P_t^{Bid} + Ke^{-r(T-t)}. \tag{22}$$

That is, an investor selling the call at the bid price C_t^{Bid}, buying the put at the ask price P_t^{Ask} , and selling the riskless bond of an amount $Ke^{-r(T-t)}$ obtains the payoff $-S_T$ at option maturity. This strategy is equivalent to selling the stock short, but without paying the shorting fee prior to the option maturity date, and yields the implied stock bid price (21). Similarly, an investor buying the call at the ask price C_t^{Ask}, selling the put at the bid price P_t^{Bid} , and buying the riskless bond of an amount $Ke^{-r(T-t)}$ obtains the payoff S_T at option maturity. This strategy is equivalent to holding the stock long, but without receiving the lending benefits prior to the option maturity date, and costs the implied stock ask price (22).[14] The implied stock bid and ask prices (21)–(22) allow us to relate our results to the documented evidence on the effects of costly short-selling on apparent put-call parity violations, which are typically measured as percentage deviations of implied stock prices from the underlying stock price (e.g., Ofek, Richardson, and Whitelaw (2004), Evans, Geczy, Musto, and Reed (2007)). Proposition 3 reports the effects of the shorting fee on the call and put prices, and on the implied stock price.

Proposition 3 (**Effects of shorting fee**). *In the economy with costly short-selling and marketmakers,*

i) The call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the shorting fee ϕ.

ii) Both the call and put bid-ask spreads are increasing in the shorting fee ϕ when $\alpha e^{(1-\alpha)\phi(T-t)} < \Phi\left(d_1\left(\phi\right)\right)/\Phi\left(d_1\left(\alpha\phi\right)\right)$.

iii) The implied stock bid and ask prices are decreasing in the shorting fee ϕ.

[14] Empirical works typically adjust the implied stock bid and ask prices (21)–(22) by also adding suitable terms to the right hand sides to account for the interim dividends and the early exercise feature of American options in their sample (e.g., Ofek, Richardson, and Whitelaw (2004), Battalio and Schultz (2011)).

Proposition 3 reveals that the call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the shorting fee ϕ (property (i)). This is because option prices (13)–(16) are weighted-averages of their respective no-arbitrage upper and lower bounds, which are the costs of and proceeds from the hedge portfolio for option sellers and buyers, respectively (Proposition 1). A higher shorting fee reduces both the cost of the hedge portfolio for a call seller as it increases the benefit of holding a share long, and also the proceeds from the hedge portfolio for a call buyer as it increases the cost of short-selling (Proposition 1, property (i)). This mechanism leads to lower call ask and bid prices. In contrast, a higher shorting fee increases both the cost of the hedge portfolio for a put seller as it increases the cost of short-selling, and also the proceeds from the hedge portfolio for a put buyer as it increases the benefit of holding a share long, leading to higher put bid and ask prices. One immediate consequence of this result is that the higher the shorting fee, the lower the call implied volatility and the higher the put implied volatility, where we here employ the standard approach of inverting the Black-Scholes formula using the option prices (13)–(16) as inputs. This finding is in line with the empirical evidence in Evans, Geczy, Musto, and Reed (2007) and Lin and Lu (2016), which demonstrate that put implied volatilities are increasing in the shorting fee.[15]

Even though the call bid and ask prices are decreasing, while those of the put are increasing, both the call and put bid-ask spreads are increasing in the shorting fee ϕ for typical options and realistic values of shorting fee and partial lending (property (ii)). The condition given in the property is equivalent to a higher shorting fee reducing the cost of the hedge portfolio for a call seller less than the proceeds from the hedge portfolio for a call buyer (see (A.39) in the Appendix). This condition arises because an increase in the shorting fee not only increases short-selling costs but also increases the benefit of holding a share long partially, and call prices decrease convexly in these costs and benefits. Hence, for relatively low levels of short-selling costs, this condition is satisfied as the call option seller is affected only partially. However, for extremely high levels of short-selling costs this relation may reverse, as the call option buyer's hedge portfolio proceeds may decrease less due to convexity. As we demonstrate in our quantitative analysis in Section 5, this condition is satisfied for option contracts with typical (e.g., short) maturities and realistic (e.g., low) values of shorting fee

[15] We demonstrate the magnitude of this effect in our quantitative analysis of Section 5.

and partial lending. We then have the result that option bid-ask spreads are increasing in the shorting fee, as empirically documented by Evans, Geczy, Musto, and Reed (2007) and Lin and Lu (2016).[16]

Turning to the implied stock prices, we see that, the higher the shorting fee ϕ, the lower the implied stock bid and ask prices (property (iii)), and hence, the higher their deviations from the underlying stock price. This is because, as we discussed earlier, the strategy that yields the implied stock bid price (21) is equivalent to selling the stock short, but without paying the shorting fee prior to the option maturity date. Hence, by no-arbitrage, the implied stock bid price must be lower than the underlying stock price. Similarly, the strategy that costs the implied stock ask price (22) is equivalent to holding the stock long, but without receiving the lending benefits prior to the option maturity date. Hence, by no-arbitrage, the implied stock ask price must be lower than the underlying stock price. A higher shorting fee being associated with higher apparent put-call parity violations is well-supported by the empirical evidence, as in Lamont and Thaler (2003), Ofek, Richardson, and Whitelaw (2004), Evans, Geczy, Musto, and Reed (2007). However, at this point it is useful to highlight that in our economy, option bid and ask prices lie within their respective no-arbitrage bounds presented in Section 2. Therefore, the implied stock prices being less than the underlying stock price does not necessarily imply arbitrage.

Proposition 4 (Effects of partial lending). *In the economy with costly short-selling and marketmakers,*

i) The call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the partial lending α.

ii) Both the call and put bid-ask spreads are decreasing in the partial lending α.

Proposition 4 reveals that the call bid and ask prices are decreasing, while those of the put are increasing in the partial lending α (property (i)). The intuition is somewhat similar

[16]Conversely, for this condition to not hold, the option maturity would need to be long, e.g., over a year, and also the shorting fees and partial lending must be unrealistically high simultaneously, e.g., higher than 40% each. However, the exchange-traded options typically have far shorter maturities and stock shorting fees are a lot lower. For instance, the median option maturity in the full sample of Ofek, Richardson, and Whitelaw (2004) is 115 days, and the typical stock in the highest shorting fee decile has a shorting fee of 6.96% in the sample of Drechsler and Drechsler (2016).

to that of the shorting fee discussed in Proposition 3. Increasing partial lending reduces the cost of the hedge portfolio for a call seller, while increasing the proceeds from the hedge portfolio for a put buyer, since it increases the benefit of holding a share long in these hedge portfolios. However, increasing partial lending has no effect on the hedge portfolio for a call buyer and the put seller since their hedge portfolios require short-selling the stock. This then decreases the call bid and ask prices and increases the put bid and ask prices since they are weighted-averages of the costs of and proceeds from these hedge portfolios (Proposition 2). Again, the immediate consequence of this result is that the higher the partial lending, the lower the call implied volatility and the higher the put implied volatility.

We see that both the call and put bid-ask spreads are decreasing in the partial lending α (property (ii)). This is in contrast to the earlier shorting fee result that option bid-ask spreads are increasing in the shorting fee for typical options (Proposition 3(ii)). This difference arises because partial lending only affects the hedge portfolio that is long in the stock and has no effect on the hedge portfolio that requires short-selling the stock, leading to an unconditional, simpler result. Therefore, a higher partial lending reduces the call ask price more than the call bid price since the hedge portfolio for a call seller requires holding a share long, resulting in a lower no-arbitrage range, and thus a lower call bid-ask spread. Similarly, a higher partial lending increases the put bid price more than the put ask bid price since the hedge portfolio for a put buyer requires holding a share long, leading to a lower no-arbitrage range, and thus a lower put bid-ask spread. To our knowledge, the opposite effect of partial lending from the shorting fee on option bid-ask spreads is a new prediction and has not been empirically explored.

As discussed in Section 3, the offsetting order arrival rates are inherently related to option liquidity. That is, the more liquid options with higher buying/selling activity are more likely to have higher offsetting order arrival rates (or, equivalently, lower expected arrival times for offsetting orders). Proposition 5 investigates the effects of the offsetting order arrival rates.

Proposition 5 (**Effects of offsetting order arrival rates**). *In the economy with costly short-selling and marketmakers,*

i) The call and put bid and ask prices are decreasing in their offsetting sell order arrival rates λ_{Cs}, λ_{Ps}, while they are increasing in their offsetting buy order arrival rates λ_{Cb}, λ_{Pb}, respectively.

ii) Both the call and put bid-ask spreads are decreasing in their offsetting order arrival rates λ_{Cs}, λ_{Cb} and λ_{Ps}, λ_{Pb}, respectively.

iii) The effects of the shorting fee and the partial lending on the call and put bid-ask spreads are decreasing in the offsetting order arrival rates λ_{Cs}, λ_{Cb}, λ_{Ps}, λ_{Pb}.

Property (i) reveals that the call and put bid and ask prices are decreasing in their offsetting sell order arrival rates λ_{Cs}, λ_{Ps}, while they are increasing in their offsetting buy order arrival rates λ_{Cb}, λ_{Pb}, respectively. This is fairly intuitive since it states that option prices are decreasing in investors' selling activity, but are increasing in buying activity. In our economy, this result is due to the fact that option bid and ask prices are given by the marketmakers' expected cost of hedging a sell and a buy order, respectively, where the expectation is taken with respect to the uncertainty about the offsetting order arrival times (Lemma 1). An increase in the arrival rates increases the probability of perfect hedging via an offsetting order, which not only costs less for hedging a buy order but also yields more proceeds from hedging a sell order as compared to the perfect hedge portfolio in the underlying stock. For instance, a higher offsetting call sell order arrival rate λ_{Cs} reduces the call ask price C_t^{Ask} because marketmakers are more likely to perfectly hedge the current buy order via an offsetting sell order at a lower cost, which in turn also reduces the call bid price C_t^{Bid} since the current value of the future call ask price is now lower in (9). Similar arguments show that a higher offsetting put sell order arrival rate λ_{Ps} reduces the put ask P_t^{Ask} and bid P_t^{Bid} prices, while a higher offsetting put buy order arrival rate increases them.

Property (ii) shows that both the call and put bid-ask spreads are decreasing in their respective offsetting order arrival rates. This result is also intuitive as it simply says that option bid-ask spreads are decreasing in liquidity (e.g., investors' buying/selling activity). In our economy, the maximum possible option bid-ask spreads are the no-arbitrage ranges in Proposition 1, and arise in the special cases of infinite expected arrival times for offsetting orders (no possibility of offsetting trades as discussed in Section 3). In contrast, the minimum possible option bid-ask spreads are zero, and arise in the special cases of zero expected arrival times for offsetting orders. As an offsetting order is expected to arrive immediately, marketmakers do not need to form the hedge portfolios, and the competition among them leads to the same ask and bid prices.

In general, the effects of short-selling costs on option prices depend on the extent of the investors' buying/selling activity, and hence options liquidity. Property (iii) shows that the extent of the effects of the shorting fee and partial lending on bid-ask spreads decrease in the investors' buying/selling activity. Hence, an increase in the shorting fee increases, while an increase in the partial lending decreases the option bid-ask spreads more for relatively illiquid options that have lower levels of offsetting order arrival rates.[17] This is intuitive as it simply says that the effects of short-selling costs are more pronounced for relatively illiquid options with lower trading activity. In our model, this occurs because marketmakers are more likely to hedge the relatively illiquid options via hedge portfolios, through which short-selling costs affect option prices directly.

5 2008 Short-Selling Ban and Quantitative Analysis

In this Section, we first apply our model to the widely-studied 2008 US short-selling ban period, during which option marketmakers were still allowed to short-sell. Consistent with empirical evidence, we first show that both the call and put bid-ask spreads and apparent put-call parity violations of banned stocks are higher than those of unbanned stocks. Second, we demonstrate the asymmetric effect of the ban on the option prices of banned stocks in that call bid prices decrease more than ask prices, and put ask prices increase more than the bid prices. We then quantify our model and demonstrate that the effects of short-selling costs on option prices are economically significant for expensive-to-short stocks. Finally, we apply our model and shed light on the behavior of option prices of the Palm stock in 2000, during which it experienced extreme short-selling and violations of the law of one price.

5.1 2008 Short-Selling Ban

In this Section, we apply our model to the September 2008 US short-selling ban period.[18] During this period the option marketmakers were exempt from the ban and were allowed to

[17]We know that both the call and put bid-ask spreads are increasing in the shorting fee ϕ for typical options (Proposition 3(ii)), and they are decreasing in the partial lending α (Proposition 4(ii)).

[18]See Battalio and Schultz (2011) for more details and relevant regulatory events for this period starting from September 19, 2008 and ending on October 8, 2008 during which a short-selling ban was imposed on nearly 800 financial stocks in the US.

short-sell to provide the marketmaking facility. Therefore, our option prices of Proposition 2 are still valid for options on both the banned and unbanned stocks during this period. The main difference between the banned and unbanned stocks was the fact that the ban led to a decrease in the partial lending while leading to a simultaneous increase in the shorting fee for the banned stocks. This is because the short-selling ban reduced the overall short-selling activity on banned stocks since the only short-sellers on them were the marketmakers and specialists. This meant that investors who were long in these stocks were more likely to lend a smaller fraction of their shares. In fact, the evidence indicates that the short-selling ban reduced (roughly halved) the short-selling activity but increased (roughly doubled) the shorting fee of banned stocks (Boehmer, Jones, and Zhang (2013), Harris, Namvar, and Phillips (2013), Kolasinski, Reed, and Thornock (2013)).[19] In light of this evidence, we take the shorting fee of the banned stocks, ϕ_{Ban}, to be twice the shorting fee of the unbanned stocks, denoted by ϕ as before. Moreover, we also take the partial lending for the banned stocks, α_{Ban} to be half of the partial lending of the unbanned stock, denoted by α as before.

In sum, these simple adjustments imply that the marketmakers were effectively paying double the rate $\phi_{Ban} = 2\phi$ for short-selling, but earning the same rate $\alpha_{Ban}\phi_{Ban} = \alpha\phi$ for holding a stock share long in their hedge portfolios of options on banned stocks as compared to options on otherwise identical unbanned stocks during the ban period, as well as to options on them before the ban.[20] Proposition 6 reports the effects of the short-selling ban on the option prices of banned and unbanned stocks during the ban period.

Proposition 6 (Effects of short-selling ban). *During the short-selling ban,*

i) The call bid and ask prices of banned stocks are lower, while the put bid and ask prices of banned stocks are higher than those of unbanned stocks.

[19] Boehmer, Jones, and Zhang (2013) report that during the ban period, shorting activity roughly halved (decreased from 21.40% to 9.96% of trading volume), while Harris, Namvar, and Phillips (2013) also document a similar magnitude for the reduction in the short interest levels of banned stocks (a decrease from roughly 7.00% to 4.00%). On the other hand, Kolasinski, Reed, and Thornock (2013) report that during the ban period, shorting fees of banned stocks roughly doubled (increased by 113% from 0.65% to 1.38%).

[20] Since the short-selling ban reduced the short-selling activity but increased the shorting fee, it was effectively a negative supply shock in the short selling and stock lending market for the banned stocks (Cohen, Diether, and Malloy (2007)). We note that doubling the shorting fee while halving the partial lending of the banned stocks allows us to demonstrate the effects of the ban clearly and in a simple fashion as Proposition 6 illustrates. Adjusting the shorting fee and partial lending exactly as in the evidence complicates the analysis unnecessarily but also leads to similar results that can be shown numerically.

ii) Both the call and put bid-ask spreads of banned stocks are higher than those of unbanned stocks.

iii) The implied stock bid and ask prices of banned stocks are lower than those of unbanned stocks.

iv) The call bid price decreases more than the ask price, while the put ask price increases more than the bid price of banned stocks.

Proposition 6 first reveals that during the short-selling ban, options on banned stocks have lower call and higher put prices as compared to those of unbanned stocks (property (i)). Proposition 6 further reveals that options on banned stocks have higher bid-ask spreads and lower implied stock prices than those of unbanned stocks (property (ii)–(iii)), consistent with empirical evidence (Battalio and Schultz (2011), Grundy, Lim, and Verwijmeren (2012), Lin and Lu (2016)). Proposition 6 also demonstrate an asymmetric effect of the short-selling ban for options on banned stocks in that their call bid prices decrease more than their ask prices, while their put ask prices increase more than their bid prices (property (iv)), also consistent with empirical evidence (Battalio and Schultz (2011)). These results arise because the short-selling ban only affects those hedge portfolios that are short in the stock, but does not affect the ones that are long in the stock since they earn the same rate per share. This reduces the proceeds from the hedge portfolios of marketmakers as call buyers but has no effect on the costs of their hedge portfolios as call sellers. Since call prices are weighted average of these costs and proceeds (Proposition 2), the short-selling ban leads to lower call bid and ask prices, higher call bid-ask spreads, and relatively higher decreases in the call bid prices. Conversely, the short-selling ban increases the costs of the hedge portfolios of marketmakers as put sellers, but has no effect on the proceeds from their hedge portfolios as put buyers. Since put prices are weighted average of these costs and proceeds, the short-selling ban leads to higher put bid and ask prices, higher put bid-ask spreads, and relatively higher increases in the put ask prices. A decrease in call prices along with an increase in put prices immediately lead to lower implied stock prices (21)–(22) which then lead to higher apparent put-call parity violations for banned stocks as compared to unbanned stocks.

5.2 Quantitative Analysis

To quantify our model, we determine the parameter values as follows. The shorting fee and partial lending values are based on the comprehensive data used in Drechsler and Drechsler (2016), who sort stocks into deciles by their shorting fee and report the average shorting fee and short interest ratios, SIR (total number of shares shorted normalized by shares outstanding), for the sample period 2004-2013. We investigate the quantitative effects of short-selling costs by considering options on a typical stock in the lowest shorting fee decile (D1) and the highest shorting fee decile (D10), henceforth expensive-to-short stocks, for which Drechsler and Drechsler report the average shorting fees to be 0.02% and 6.96%, respectively. We next take the ratio of a stock's short interest to long interest (the short interest plus outstanding shares) to be an observable proxy for its partial lending parameter α. This ratio is a plausible proxy since it gives the fraction of aggregate long position lent to short-sellers. Normalizing by the outstanding shares gives this ratio in terms of only the short interest ratio as $SIR/(1 + SIR)$. Moreover, since lenders are mainly institutions in reality (see, Reed (2013)), we further refine this measure by considering the short interest ratios normalized by institutional ownership, denoted by SIR_{IO}. These are readily provided for each decile in Drechsler and Drechsler, who report the values of 4.5% and 26.5%, for the lowest (D1) and the highest (D10) shorting fee deciles, respectively. Applying these values to $SIR_{IO}/(1 + SIR_{IO})$, we obtain the partial lending parameter values for these deciles as 4.31% and 20.95%, respectively.[21]

For the securities market parameter values, we take the interest rate to be the average 3-month T-bill rate for the sample period of Drechsler and Drechsler (2016) which is 1.80%. We set the stock price as the reported average stock price of 32.20 in Ofek, Richardson, and Whitelaw (2004).[22] The return volatility of the stock is set to 40% as in Jensen and Pedersen (2016). For option specific parameter values, we consider varying moneyness (e.g.,

[21]We note that in Drechsler and Drechsler (2016), the sample average of the short interest ratio is 9.02%, which would imply an average partial lending value of 8.27% for a typical stock according to our formula above. This value is comparable to the average fraction of outstanding shares actually lent out in the US, 8.91%, in the sample of Saffi and Sigurdsson (2010) (discussed in Section 2).

[22]Other empirical works studying the effects of short-selling costs on options also document similar values for the average stock price in their samples (e.g. it is 30.76 in Battalio and Schultz (2011)). Importantly, our main results do not vary much with any particular value of the stock price as we use the same value for the typical stock in D1 and D10.

Name	Symbol	Value
Shorting fee	ϕ	$\{0.02\%, 6.96\%\}$ varying
Partial lending	α	$\{4.31\%, 20.95\%\}$ varying
Offsetting order arrival rates	$\lambda_{Cs}, \lambda_{Cb}, \lambda_{Cb}, \lambda_{Pb}$	2.77
Stock price	S_t	32.20
Stock return volatility	σ	40.00%
Riskless interest rate	r	1.80%
Option moneyness	K/S_t	$\{0.90, 1.00, 1.10\}$ varying
Option time to maturity	$T-t$	0.25

Table 1: **Parameter values.** This table reports the parameter values used in our quantitative analysis. The determination of these values is presented in the text.

the ratio of option strike to stock price, K/S_t) levels of $0.90, 1.00, 1.10$ to demonstrate the varying effects of short-selling costs across option moneyness. Option time-to-maturity is taken to be 0.25 (3 months) which is well within the reported average option maturities in the samples of empirical works that we compare our results to.[23] Finally, we determine the offsetting call sell order arrival rates by giving equal weights to both ways of hedging in our model, hedging via an offsetting order and via a hedge portfolio. For instance, giving a probability of 0.5 to there being no arrival of an offsetting call sell order by the maturity date, $1-\int_t^T dF_{Cs}(u) = e^{-\lambda_{Cs}(T-t)}$ yields the value for the offsetting call sell order arrival rate as 2.77, which is also the value of all the other arrival rates.[24] This procedure leads to the parameter values in Table 1.

Table 2 reports the quantitative effects of short-selling costs on our option prices of Proposition 2. We consider a call option (Panel (a)) and a put option (Panel (b)) of a typical stock in the lowest (D1) and the highest (D10) shorting fee deciles for three different

[23]For instance, the median option maturity in the full sample of Ofek, Richardson, and Whitelaw (2004) is 115 days. We also provide Tables 5–6 in Appendix B for the effects of short-selling costs on options with a shorter maturity of 1.5 months and a longer maturity of 4.5 months, respectively.

[24]We recognize that the arrival rates for call and put buy and sell orders may be different since they are inherently linked to option buying/selling activity, which may differ across options as shown by Lakonishok, Lee, Pearson, and Poteshman (2007). However, keeping the same value for all arrival rates allows us to more clearly compare the quantitative effects of short-selling costs.

option moneyness levels. We also report the percentage differences of mid-points of bid and ask prices (denoted by C_t^{Mid} and P_t^{Mid} for call and put options, respectively) from the standard Black-Scholes in the relative change column.

Table 2 reveals that our model implies a significantly lower call and a higher put bid and ask prices for the typical stock in D10, as compared to those for the typical stock in D1. In particular, for the typical stock in D10, its at-the-money (ATM) call mid-point price is 6.80% lower (-6.82% vs -0.02%), while that of the put is 6.28% higher (6.30% vs 0.02%) than the corresponding ones in D1. We see that these effects are stronger for out-of-the-money call and put options, being 8.17% lower and 8.15% higher for option moneyness of 1.10 and 0.90, respectively. Table 2 also quantifies the relative bid-ask spread by reporting the ratio of the bid-ask spread to the mid-point prices. We see that the typical stock in D10, has a 2.35% higher ATM call bid-ask spread as compared to the spread of the typical stock in D1. For the ATM put this difference is 1.95%. Again, these effects are stronger for out-of-the-money options. We also see that the typical stock in D10 has a 2.81% lower (37.18% vs 39.99) ATM call implied volatility as compared to the implied volatility of the typical stock in D1. However, for the ATM put, the typical stock in D10 has a 2.45% higher (42.46 vs 40.01%) implied volatility compared to the implied volatility of the typical stock in D1.

Finally, substituting the at-the-money option prices in Table 2 into (21)–(22) yields the implied stock bid and ask prices for the typical stock in D1 to be the same as the underlying stock price. However, this procedure yields the implied stock bid and ask prices to be 31.81 and 31.92, respectively, for the typical stock in D10. In terms of percentage deviation these values imply a 1.04% lower implied stock mid-price from the underlying stock price. This magnitude is within the documented range in Evans, Geczy, Musto, and Reed (2007) who report an average deviation of 0.36% and the 90th percentile deviation of 1.40% in their sample.[25]

We now apply our model to the option prices of the Palm stock during its IPO in March, 2000. This event was notable since it was a prime example of apparent violations of the law

[25]Similarly, Ofek, Richardson, and Whitelaw (2004) find that a one standard deviation (2.77%) increase in the shorting fee leads to a 0.67% lower implied stock price as compared to the underlying stock price in their sample. For this magnitude of an increase in the shorting fee, our model implies a comparable 0.60% lower implied stock price after also adjusting for their sample average maturity. We note that the implied stock prices are stable and do not vary much in option moneyness, and therefore it is sufficient to only consider the at-the-money option prices to derive the implied stock prices as we do here.

Panel (a): Call option

Shorting fee decile	Option moneyness	BS price	Bid price	Ask price	Relative change	Bid-Ask spread	Relative spread	Implied volatility
	$\frac{K}{S_t}$	C_t^{BS}	C_t^{Bid}	C_t^{Ask}	$\frac{C_t^{Mid}-C_t^{BS}}{C_t^{BS}}$	$C_t^{Ask}-C_t^{Bid}$	$\frac{C_t^{Ask}-C_t^{Bid}}{C_t^{Mid}}$	$\tilde{\sigma}_{t,C^{Mid}}$
D1	1.10	1.43	1.43	1.43	-0.02%	0.00	0.01%	39.99%
D10		1.43	1.29	1.33	-8.19%	0.04	2.86%	38.05%
D1	1.00	**2.63**	**2.63**	**2.63**	**-0.02%**	**0.00**	**0.01%**	**39.99%**
D10		**2.63**	**2.42**	**2.48**	**-6.82%**	**0.06**	**2.36%**	**37.18%**
D1	0.90	4.46	4.46	4.46	-0.01%	0.00	0.01%	39.99%
D10		4.46	4.18	4.26	-5.50%	0.08	1.88%	35.19%

Panel (b): Put option

	$\frac{K}{S_t}$	P_t^{BS}	P_t^{Bid}	P_t^{Ask}	$\frac{P_t^{Mid}-P_t^{BS}}{P_t^{BS}}$	$P_t^{Ask}-P_t^{Bid}$	$\frac{P_t^{Ask}-P_t^{Bid}}{P_t^{Mid}}$	$\tilde{\sigma}_{t,P^{Mid}}$
D1	1.10	4.49	4.49	4.49	0.01%	0.00	0.01%	40.01%
D10		4.49	4.67	4.74	4.89%	0.07	1.53%	43.61%
D1	1.00	**2.49**	**2.49**	**2.49**	**0.02%**	**0.00**	**0.01%**	**40.01%**
D10		**2.49**	**2.62**	**2.67**	**6.30%**	**0.05**	**1.96%**	**42.46%**
D1	0.90	1.11	1.11	1.11	0.02%	0.00	0.01%	40.00%
D10		1.11	1.19	1.22	8.17%	0.03	2.51%	41.74%

Table 2: **Quantitative effects of costly short-selling on option prices.** This table reports the effects of costly short-selling for a call option (Panel (a)) and a put option (Panel (b)) on a typical stock in the lowest (D1) and the highest (D10) shorting fee decile in Drechsler and Drechsler (2016) for three different option moneyness levels. C_t^{Mid} and P_t^{Mid} denote the mid-point prices of the call and put, e.g., $C_t^{Mid} = 0.5(C_t^{Ask} + C_t^{Bid})$ and $P_t^{Mid} = 0.5(P_t^{Ask} + P_t^{Bid})$. Implied volatilities in the last columns are obtained by employing the standard approach of inverting the Black-Scholes formula using the mid-point option prices as inputs. All parameter values are as in Table 1.

of one price (Lamont and Thaler (2003)). In particular, there was a long-lasting mispricing of Palm relative to its parent company 3Com in the sense that the subsidiary Palm was worth more than its parent company 3Com. This long-lasting mispricing of 3Com/Palm was often attributed to the extreme short-selling costs of Palm. In particular, its shorting fee was reported to be around 35% during this period (D'Avolio (2002)), and its short interest after the IPO in March was 19.4%, then increased to 44.9% in April, and to 70% in May, and peaked at 147.6% in July (Lamont and Thaler (2003)).[26]

We demonstrate the effects of short-selling costs on Palm option prices by comparing our prices to those in Lamont and Thaler (2003), who provide prices of at-the-money Palm options on March 17, 2000 for three different maturities, May ($T - t = 0.17$), August ($T - t = 0.42$) and November ($T - t = 0.67$). We take the shorting fee to be its reported value of 35%. Then we use the (approximate) average short interest ratio for the August maturity option, 70% to back out our partial lending parameter value as before and obtain $0.70/1.70 = 41.18\%$. For the securities market parameter values, we follow Lamont and Thaler (2003) and set the interest rate as the 3-month LIBOR rate of 6.21% to price May options, and the 6-month LIBOR rate of 6.41% to price August and November options. We set the stock price as the reported Palm stock price on March 17, 2000 of 55.25, which is also the strike price for the at-the-money options considered. The return volatility of the Palm stock is set to its average realized volatility during the life of the mid-maturity option expired in August, 104.6%.[27] Finally, we again give equal weights to both ways of hedging for August maturity option, hedging via an offsetting order and via a hedge portfolio. This yields the value for all the offsetting order arrival rates as 1.66, which is also kept the same for the May and November maturity options. Using these parameter values, we now quantify the effects of costly short-selling on Palm options and present our results in Table 3 for at-the-money call and put options, as well as for the implied stock prices and their percentage deviations from the underlying stock price for three different option maturity dates.

Table 3 reveals that Palm option prices displayed significant apparent put-call parity violations, in the sense that put prices were higher than call prices (which should not happen for at-the-money options), and the implied stock prices were significantly lower than the underlying stock price. In particular, the evidence indicates that for the mid-maturity options expiring in August, call bid and ask prices of 9.25 and 10.75 were significantly less than the put bid and ask prices of 17.25 and 19.25, respectively. Our model also generates this feature by yielding lower call bid and ask prices of 11.54 and 12.18 than put bid and ask prices of

[26] Since both the short-selling activity and the shorting fee increased, there was effectively a positive demand shock in the short selling and stock lending market of the Palm stock as described in (Cohen, Diether, and Malloy (2007)).

[27] We estimated the return volatility of Palm in a standard way using the standard CRSP data. Considering the shorter maturity May or longer maturity November also give similar very high volatility values.

Palm Options on March 17, 2000

	Option maturity	Call		Put		Implied stock			
		Bid	Ask	Bid	Ask	Bid		Ask	
	$T-t$	C_t^{Bid}	C_t^{Ask}	P_t^{Bid}	P_t^{Ask}	$\tilde{S}_t^{Bid}$		$\tilde{S}_t^{Ask}$	
Evidence		5.75	7.25	10.63	12.63	47.55	-13.93%	51.05	-7.60%
Our Model	May	8.02	8.60	9.73	10.20	52.51	-4.96%	53.56	-3.06%
Black-Scholes		9.60	9.60	9.01	9.01	55.25	0.00%	55.25	0.00%
Evidence		9.25	10.75	17.25	19.25	43.57	-21.14%	47.07	-14.81%
Our Model	Aug	**11.54**	**12.18**	**15.54**	**15.98**	**49.36**	**-10.66%**	**50.43**	**-8.72%**
Black-Scholes		15.15	15.15	13.70	13.70	55.25	0.00%	55.25	0.00%
Evidence		10.00	11.50	21.63	23.63	39.12	-29.19%	42.62	-22.86%
Our Model	Nov	13.52	13.95	19.56	19.84	46.62	-15.62%	47.32	-14.35%
Black-Scholes		19.06	19.06	16.75	16.75	55.25	0.00%	55.25	0.00%

Table 3: **Quantitative effects of costly short-selling on Palm options.** This table reports the effects of extreme short-selling on the Palm options and implied stock prices on March 17, 2000 for three different option maturity dates, May 20, August 19 and November 18, 2000. The values in the Evidence rows are from Lamont and Thaler (2003) (Table 6, p. 256). The parameter values used in our model and the Black-Scholes model are as discussed in text: $\phi = 35\%$, $\alpha = 41.18\%$, $r = 6.21\%$ (May), $r = 6.41\%$ (Aug, Nov), $S_t = K = 55.25$, $\sigma = 104.6\%$, and $\lambda_{Cs} = \lambda_{Cb} = \lambda_{Cb} = \lambda_{Pb} = 1.66$.

15.54 and 15.98, a feature not possible in the standard Black-Scholes model. In terms of the deviation from the underlying stock price, we see that the implied stock bid price was 21.14% and ask price 14.81% lower than the underlying stock price. The option prices implied by our model are able to generate roughly half of this deviation as they imply 10.66% (bid) and 8.72% (ask) lower prices. As Lamont and Thaler (2003) also highlight, high levels of short-selling costs were only part of the story as there were several other extreme risks and costs for short-sellers of the Palm stock during that time (e.g., search costs, uncertainties about collateral levels, shorting fee and early recall of the shares by lenders). Nevertheless, our model demonstrates that, for mid and long maturity options, roughly half of the price deviations could be due to the costly short-selling implying that the combined effects of all the other risks, costs and considerations could amount to the remaining half.

6 Conclusion

In this paper, we have provided an analysis of option prices in the presence of costly short-selling by adopting the classic Black-Scholes framework. We have shown that standard no-arbitrage restrictions alone cannot determine option prices but only lead to lower and upper bounds for them with costly short-selling. With option marketmakers present, we then obtained unique option bid and ask prices in closed-form, representing the marketmakers' expected cost of hedging and preserving the well-known properties of the Black-Scholes prices.

Consistently with empirical evidence, we have shown that bid-ask spreads of typical options and apparent put-call parity violations are increasing in the shorting fee. We have also found that option bid-ask spreads are decreasing in the partial lending, and the effects of costly short-selling are more pronounced for relatively illiquid options. Moreover, by applying our model to the recent 2008 short-selling ban period, we have demonstrated the asymmetric effect of the ban on the prices of options on banned stocks, whose bid-ask spreads and apparent put-call parity violations are higher than those for unbanned stocks, consistently with empirical evidence. Finally, our quantitative analysis has demonstrated that the effects of costly short-selling on option prices are economically significant for expensive-to-short stocks.

As we highlighted in Remarks 1–2, so as to not unnecessarily confound or complicate our analysis, we did not consider time-variation in short-selling costs and other potential features of option marketmaking. We leave these considerations and other relevant issues for future research.

Appendix A: Proofs

Proof of Proposition 1. To determine the option prices that admit no-arbitrage we first find the cost of the perfect hedge portfolio, the self-financing portfolio in the underlying stock and riskless bond that perfectly hedges (offsets) the option seller and buyer's payoff at its maturity. We then use standard arguments to determine the range of option prices that admit no-arbitrage. For a given option payoff, we denote its time-t hedge portfolio cost by V_t, the number of units in the riskless bond by β_t, and the number of shares in the underlying stock by θ_t.

We first consider the call option seller's payoff $-\max\{S_T - K, 0\}$. The portfolio that perfectly hedges this payoff at its maturity must have $V_T = \max\{S_T - K, 0\}$. To determine the hedge portfolio cost V_t for all $t < T$, we conjecture that the hedge portfolio is always long in the underlying stock, $\theta_t > 0$ for all $t \leq T$. In this case, the fraction α of the long position is lent to short-sellers. We decompose the cost of the hedge portfolio as

$$V_t = \beta_t B_t + \theta_t S_t = \beta_t B_t + (1-\alpha)\,\theta_t S_t + \alpha\theta_t S_t, \tag{A.1}$$

where the last term $\alpha\theta_t S_t$ is the total amount lent to short sellers, which in addition to the stock capital gains also earns the shorting fee ϕ. Hence, the dynamics of the self-financing hedge portfolio is given by

$$\begin{aligned} dV_t &= \beta_t dB_t + (1-\alpha)\,\theta_t dS_t + \alpha\theta_t\,(dS_t + \phi S_t dt) \\ &= rV_t dt + (\mu - r + \alpha\phi)\,\theta_t S_t dt + \sigma\theta_t S_t d\omega_t, \end{aligned} \tag{A.2}$$

where the second equality follows by substituting the bond and stock dynamics (1)–(2) and $\beta_t B_t$ from (A.1), and rearranging. We observe that (A.2) is the dynamics of the self-financing hedge portfolio in the Black-Scholes economy where the underlying stock pays a continuous dividend at a constant rate $\alpha\phi$. Since $V_T = \max\{S_T - K, 0\}$, standard valuation arguments (e.g., McDonald (2006)) yield the cost of the hedge portfolio to be $V_t = C_t^{BS}(\alpha\phi)$, where $C_t^{BS}(q)$ denotes the standard Black-Scholes call price adjusted for the constant dividend yield q and is given by (5). Lastly, we confirm our conjecture that the hedge portfolio is always long in the underlying stock by showing

$$\theta_t = \frac{\partial}{\partial S_t} C_t^{BS}(\alpha\phi) = e^{-\alpha\phi(T-t)} \Phi(d_1(\alpha\phi)) > 0, \tag{A.3}$$

where $\Phi(.)$ is the standard normal cumulative distribution function and $d_1(q)$ is as in (7).

We next consider the call option buyer's payoff $\max\{S_T - K, 0\}$. The portfolio that perfectly hedges this payoff at its maturity must have $V_T = -\max\{S_T - K, 0\}$. To determine the hedge portfolio cost V_t for all $t < T$, we conjecture that the hedge portfolio is always short in the underlying stock, $\theta_t < 0$ for all $t \leq T$. In this case, the cost of the hedge portfolio is

$$V_t = \beta_t B_t + \theta_t S_t + M_t, \tag{A.4}$$

where the last term M_t denotes the total amount collateralized, and hence cannot be invested in other securities, and is given by $M_t = -\theta_t S_t > 0$. For the short-seller this account earns the rebate rate $r - \phi$, implying its dynamics as $dM_t = (r - \phi) M_t dt$. Hence, the dynamics of the hedge portfolio cost is given by

$$\begin{aligned} dV_t &= \beta_t dB_t + \theta_t dS_t + dM_t \\ &= rV_t dt + (\mu - r + \phi)\theta_t S_t dt + \sigma\theta_t S_t d\omega_t, \end{aligned} \tag{A.5}$$

where the second equality follows by substituting the bond and stock dynamics (1)–(2) and $\beta_t B_t$ from (A.4), and rearranging. This is the dynamics of the self-financing hedge portfolio in the Black-Scholes economy where the underlying stock pays a continuous dividend at a constant rate ϕ. Since $V_T = -\max\{S_T - K, 0\}$, standard valuation arguments yield the cost of the hedge portfolio to be $V_t = -C_t^{BS}(\phi)$, where a negative cost means proceeds, a positive amount $C_t^{BS}(\phi)$ that investors receive. Lastly, we confirm our conjecture that the hedge portfolio is always short in the underlying stock by showing

$$\theta_t = \frac{\partial}{\partial S_t}\left(-C_t^{BS}(\phi)\right) = -e^{-\phi(T-t)}\Phi(d_1(\phi)) < 0. \tag{A.6}$$

Having determined the cost of the hedge portfolio for the call option buyer and seller, we now show that a call option price, C_t, admits no-arbitrage if and only if the double inequality

(3) in Proposition 1 is satisfied. To see this, suppose by contradiction that the call option were trading at a price $C_t^{BS}(\alpha\phi) < C_t$. Then selling the call option at the price C_t and forming the hedge portfolio at the cost $C_t^{BS}(\alpha\phi)$ would lead to a zero payoff at the option maturity date. However, this strategy has a positive initial profit $C_t - C_t^{BS}(\alpha\phi)$, hence this option price admits arbitrage. Now, suppose by contradiction that the call option were trading at a price $C_t < C_t^{BS}(\phi)$. Then buying the call option at the price C_t and forming the hedge portfolio by receiving $C_t^{BS}(\phi)$ would lead to a zero payoff at the option maturity date. However, this strategy has a positive initial profit $C_t^{BS}(\phi) - C_t$, hence this option price also admits arbitrage. On the other hand, if the call price satisfies the double inequality (3), then it admits no-arbitrage because selling or buying the option and perfectly hedging it at its maturity can at most lead to a zero initial profit.

We now consider the put option seller's payoff $-\max\{K - S_T, 0\}$. The portfolio that perfectly hedges this payoff at its maturity must have $V_T = \max\{K - S_T, 0\}$. To determine the hedge portfolio cost we conjecture that the hedge portfolio is always short in the underlying stock. Following the same steps as in the call option buyer's payoff above leads to the dynamics (A.5), hence standard valuation arguments yield the cost of the hedge portfolio to be $V_t = P_t^{BS}(\phi)$, where $P_t^{BS}(q)$ denotes the standard Black-Scholes put price adjusted for the constant dividend yield q and is given by (6). Lastly, we confirm our conjecture that the hedge portfolio is always short in the underlying stock by showing

$$\theta_t = \frac{\partial}{\partial S_t} P_t^{BS}(\phi) = -e^{-\phi(T-t)}\Phi(-d_1(\phi)) < 0. \tag{A.7}$$

We next consider the put option buyer's payoff $\max\{K - S_T, 0\}$. The portfolio that perfectly hedges this payoff at its maturity must have $V_T = -\max\{K - S_T, 0\}$. To determine the hedge portfolio cost we conjecture that the hedge portfolio is always long in the underlying stock. Following the same steps as in the call option seller's payoff above leads to the dynamics (A.2), hence standard valuation arguments yield the cost of the hedge portfolio to be $V_t = -P_t^{BS}(\alpha\phi)$, where again a negative cost means proceeds. Lastly, we confirm our conjecture that the hedge portfolio is always long in the underlying stock by showing

$$\theta_t = \frac{\partial}{\partial S_t}\left(-P_t^{BS}(\alpha\phi)\right) = e^{-\alpha\phi(T-t)}\Phi(-d_1(\alpha\phi)) > 0. \tag{A.8}$$

Going through the same steps as in the call option case shows that a put option price, P_t, admits no-arbitrage if and only if the double inequality (4) in Proposition 1 is satisfied.

The unique no-arbitrage call and put prices in the special case of full lending (8) follow immediately by substituting $\alpha = 1$ into the inequalities (3)–(4).

Property (i) that the call price upper bound is decreasing in both the shorting fee and partial lending, while its lower bound is decreasing in the shorting fee but not dependent on the partial lending follows from the partial derivative of the standard Black-Scholes call price with respect to the dividend yield (e.g., McDonald (2006))

$$\frac{\partial}{\partial q} C_t^{BS}(q) = -(T-t)\, S_t e^{-q(T-t)} \Phi\left(d_1(q)\right) < 0, \tag{A.9}$$

which implies

$$\begin{aligned} &\frac{\partial}{\partial \phi} C_t^{BS}(\phi) < 0, && \frac{\partial}{\partial \phi} C_t^{BS}(\alpha\phi) < 0, \\ &\frac{\partial}{\partial \alpha} C_t^{BS}(\phi) = 0, && \frac{\partial}{\partial \alpha} C_t^{BS}(\alpha\phi) < 0. \end{aligned} \tag{A.10}$$

Property (ii) that the put price upper bound is increasing in the shorting fee but is not dependent on the partial lending, while its lower bound is increasing in both the shorting fee and partial lending follows from the partial derivative of the standard Black-Scholes put price with respect to the dividend yield (e.g., McDonald (2006))

$$\frac{\partial}{\partial q} P_t^{BS}(q) = (T-t)\, S_t e^{-q(T-t)} \Phi\left(-d_1(q)\right) > 0, \tag{A.11}$$

which implies

$$\begin{aligned} &\frac{\partial}{\partial \phi} P_t^{BS}(\phi) > 0, && \frac{\partial}{\partial \phi} P_t^{BS}(\alpha\phi) > 0, \\ &\frac{\partial}{\partial \alpha} P_t^{BS}(\phi) = 0, && \frac{\partial}{\partial \alpha} P_t^{BS}(\alpha\phi) > 0. \end{aligned} \tag{A.12}$$

Property (iii) that the call prices are lower, while the put prices are higher than the corresponding prices in the Black-Scholes is immediate from properties (i)–(ii) since the Black-Scholes prices are obtained when these bounds are evaluated at $\phi = 0$. $\square$

Proof of Lemma 1. We first derive the expected cost of hedging representations for the call bid and ask prices in detail, and then for the put prices relying on similar arguments. At time t, the call bid price is set by the competitive marketmakers such that buying a call option at the bid price C_t^{Bid} and forming the hedge portfolio and receiving the proceeds $C_t^{BS}(\phi)$ (Proposition 1) yields zero expected profit. The difference $C_t^{BS}(\phi) - C_t^{Bid}$ is invested in the riskless bond earning the interest rate r. If there is no offsetting call buy order by the maturity date T, the marketmaker's profit from this trade at maturity becomes the amount $[C_t^{BS}(\phi) - C_t^{Bid}]\, e^{r(T-t)}$, which has a current value, $\Pi_{T,C^{Bid}}$, of

$$\Pi_{T,C^{Bid}} \equiv C_t^{BS}(\phi) - C_t^{Bid}. \tag{A.13}$$

If an offsetting call buy order arrives at a subsequent time $u < T$, the marketmaker sells a call option at an ask price C_u^{Ask}, and liquidates the hedge portfolio at a value of $-C_u^{BS}(\phi)$. This leads to the marketmaker's profit from this trade at time u as

$$\left[C_t^{BS}(\phi) - C_t^{Bid}\right] e^{r(u-t)} + \left[C_u^{Ask} - C_u^{BS}(\phi)\right],$$

which has a current value, $\Pi_{u,C^{Bid}}$, of

$$\Pi_{u,C^{Bid}} \equiv \left[C_t^{BS}(\phi) - C_t^{Bid}\right] + V_t\left[C_u^{Ask} - C_u^{BS}(\phi)\right], \tag{A.14}$$

where $V_t\,[C_u^{Ask} - C_u^{BS}(\phi)]$ is the current value of the time-u random payoff $C_u^{Ask} - C_u^{BS}(\phi)$ which is yet to be determined. Therefore, the marketmaker's expected profit from buying a call option, $\Pi_{C^{Bid}}$, becomes

$$\Pi_{C^{Bid}} \equiv \int_t^T \Pi_{u,C^{Bid}} dF_{Cb}(u) + \Pi_{T,C^{Bid}}\Big(1 - \int_t^T dF_{Cb}(u)\Big), \tag{A.15}$$

where F_{Cb} is the distribution function of the offsetting call buy order arrival time. Substituting (A.13)–(A.14) into (A.15) and rearranging yields the expected profit

$$\Pi_{C^{Bid}} = \left[C_t^{BS}(\phi) - C_t^{Bid}\right] + \int_t^T V_t\left[C_u^{Ask} - C_u^{BS}(\phi)\right] dF_{Cb}(u). \tag{A.16}$$

By equating the expected profit (A.16) to zero we back out the call bid price as

$$C_t^{Bid} = C_t^{BS}(\phi) + \int_t^T V_t \left[C_u^{Ask} - C_u^{BS}(\phi)\right] dF_{Cb}(u).$$

Adding and subtracting $\int_t^T C_t^{BS}(\phi)\, dF_{Cb}(u)$ to the right hand side gives the expected cost representation for the call bid price in (9).

Similarly, at time t, the call ask price is set by the competitive marketmakers such that selling a call option at the ask price C_t^{Ask} and forming the hedge portfolio at a cost $C_t^{BS}(\alpha\phi)$ (Proposition 1) yields zero expected profit. The difference $C_t^{Ask} - C_t^{BS}(\alpha\phi)$ is again invested in the riskless bond. If there is no offsetting call sell order by the option maturity date T, the marketmaker's profit from this trade at maturity becomes the amount $[C_t^{Ask} - C_t^{BS}(\alpha\phi)]\, e^{r(T-t)}$, which has a current value, $\Pi_{T,C^{Ask}}$, of

$$\Pi_{T,C^{Ask}} \equiv C_t^{Ask} - C_t^{BS}(\alpha\phi). \tag{A.17}$$

If an offsetting call sell order arrives at a subsequent time $u < T$, the marketmaker buys a call option at a bid price C_u^{Bid}, and liquidates the hedge portfolio at a value of $C_u^{BS}(\alpha\phi)$. This leads to the marketmaker's profit from this trade at time u as

$$\left[C_t^{Ask} - C_t^{BS}(\alpha\phi)\right] e^{r(u-t)} + C_u^{BS}(\alpha\phi) - C_u^{Bid},$$

which has a current value, $\Pi_{u,C^{Ask}}$, of

$$\Pi_{u,C^{Ask}} \equiv \left[C_t^{Ask} - C_t^{BS}(\alpha\phi)\right] + V_t\left[C_u^{BS}(\alpha\phi) - C_u^{Bid}\right], \tag{A.18}$$

where $V_t[C_u^{BS}(\alpha\phi) - C_u^{Bid}]$ is the current value of the time-u random payoff $C_u^{BS}(\alpha\phi) - C_u^{Bid}$ which is yet to be determined. Therefore, the marketmaker's expected profit from selling of a call option, $\Pi_{C^{Ask}}$, becomes

$$\Pi_{C^{Ask}} \equiv \int_t^T \Pi_{u,C^{Ask}} dF_{Cs}(u) + \Pi_{T,C^{Ask}}\Big(1 - \int_t^T dF_{Cs}(u)\Big), \tag{A.19}$$

where F_{Cs} is the distribution function of the offsetting call sell order arrival time. Substituting

(A.17)–(A.18) into (A.19) and rearranging yields the expected profit

$$\Pi_{C^{Ask}} = \left[C_t^{Ask} - C_t^{BS}(\alpha\phi)\right] + \int_t^T V_t \left[C_u^{BS}(\alpha\phi) - C_u^{Bid}\right] dF_{Cs}(u). \tag{A.20}$$

By equating the expected profit (A.20) to zero we back out the call ask price as

$$C_t^{Ask} = C_t^{BS}(\alpha\phi) - \int_t^T V_t \left[C_u^{BS}(\alpha\phi) - C_u^{Bid}\right] dF_{Cs}(u).$$

Adding and subtracting $\int_t^T C_t^{BS}(\alpha\phi)\, dF_{Cs}(u)$ to the right hand side of the above equation gives the expected cost representation for the call ask price in (10).

For the expected cost of hedging representations for the put prices, going through the same steps as in the call option case yields the put bid and ask prices as

$$\begin{aligned} P_t^{Bid} &= P_t^{BS}(\alpha\phi) + \int_t^T V_t \left[P_u^{Ask} - P_u^{BS}(\alpha\phi)\right] dF_{Pb}(u), \\ P_t^{Ask} &= P_t^{BS}(\phi) - \int_t^T V_t \left[P_u^{BS}(\phi) - P_u^{Bid}\right] dF_{Ps}(u), \end{aligned}$$

and adding and subtracting $\int_t^T P_t^{BS}(\alpha\phi)\, dF_{Pb}(u)$ and $\int_t^T P_t^{BS}(\phi)\, dF_{Ps}(u)$, respectively, to the right hand sides of the above equations give the expected cost representations for the put bid and ask prices in (11)–(12). □

Proof of Proposition 2. To determine the call bid and ask prices, we first conjecture the functional forms for them. Then, using these functional forms we determine the current values of random payoffs in the expected cost of hedging representations for the prices in Lemma 1. Finally, by solving the resulting system of equations, we obtain the option prices in closed-form and verify our conjectured functional forms.

We conjecture that the call bid and ask prices take the forms

$$C_t^{Bid} = \left(1 - \mathrm{w}_{t,C^{Bid}}\right) C_t^{BS}(\alpha\phi) + \mathrm{w}_{t,C^{Bid}} C_t^{BS}(\phi), \tag{A.21}$$

$$C_t^{Ask} = \mathrm{w}_{t,C^{Ask}} C_t^{BS}(\alpha\phi) + \left(1 - \mathrm{w}_{t,C^{Ask}}\right) C_t^{BS}(\phi), \tag{A.22}$$

for all $t \le T$ and the deterministic weight processes $\mathrm{w}_{t,C^{Bid}}$, $\mathrm{w}_{t,C^{Ask}}$ to be identified later.

Given our conjecture, the time-u random payoff of the call bid price (9) becomes

$$C_u^{Ask} - C_u^{BS}(\phi) = \mathrm{w}_{u,C^{Ask}} C_u^{BS}(\alpha\phi) + \mathrm{w}_{u,C^{Ask}} \left(-C_u^{BS}(\phi)\right). \tag{A.23}$$

The current value of this random payoff is given by the amount required at time-t to form a self-financing portfolio in the underlying stock and riskless bond to obtain this payoff at time $u \geq t$. For this, we consider two positions, where one is long and the other is short in the underlying stock for all $u \geq t$. The first position consists of $\mathrm{w}_{u,C^{Ask}}$ units in the call option seller's hedge portfolio that is long in the stock (A.3) where $\mathrm{w}_{u,C^{Ask}}$ is a positive constant. This position has a value of $\mathrm{w}_{u,C^{Ask}} C_u^{BS}(\alpha\phi)$ at time-u with its current value given by

$$V_t\left[\mathrm{w}_{u,C^{Ask}} C_u^{BS}(\alpha\phi)\right] = \mathrm{w}_{u,C^{Ask}} C_t^{BS}(\alpha\phi), \tag{A.24}$$

since this is the amount required at time-t for a self-financing portfolio to obtain the payoff $\mathrm{w}_{u,C^{Ask}} C_u^{BS}(\alpha\phi)$ at time u. Similarly, the second position consists of $\mathrm{w}_{u,C^{Ask}}$ units in the call option buyer's hedge portfolio that is short in the stock (A.6). This position has a value of $\mathrm{w}_{u,C^{Ask}}\left(-C_u^{BS}(\phi)\right)$ at time-u with its current value given by

$$V_t\left[\mathrm{w}_{u,C^{Ask}}\left(-C_u^{BS}(\phi)\right)\right] = \mathrm{w}_{u,C^{Ask}}\left(-C_t^{BS}(\phi)\right). \tag{A.25}$$

Summing (A.24) and (A.25) gives the current value of the random payoff (A.23) as

$$V_t\left[C_u^{Ask} - C_u^{BS}(\phi)\right] = \mathrm{w}_{u,C^{Ask}} C_t^{BS}(\alpha\phi) + \mathrm{w}_{u,C^{Ask}}\left(-C_t^{BS}(\phi)\right).$$

Substituting this into the call bid price representation (9) and rearranging gives

$$C_t^{Bid} = \left[\int_t^T \mathrm{w}_{u,C^{Ask}} dF_{Cb}(u)\right] C_t^{BS}(\alpha\phi) + \left[1 - \int_t^T \mathrm{w}_{u,C^{Ask}} dF_{Cb}(u)\right] C_t^{BS}(\phi). \tag{A.26}$$

Similarly, given our conjecture, the time-u random payoff of the call ask price (10) becomes

$$C_u^{BS}(\alpha\phi) - C_u^{Bid} = \mathrm{w}_{u,C^{Bid}} C_u^{BS}(\alpha\phi) + \mathrm{w}_{u,C^{Bid}}\left(-C_u^{BS}(\phi)\right). \tag{A.27}$$

To determine the current value of this random payoff, we again consider two positions, where one is long and the other is short in the underlying stock for all $u \geq t$. The first position consists of $\mathrm{w}_{u,C^{Bid}}$ units in the call option seller's hedge portfolio that is long in the stock. This position has a value of $\mathrm{w}_{u,C^{Bid}} C_u^{BS}(\alpha\phi)$ at time-u with its current value given by

$$V_t\left[\mathrm{w}_{u,C^{Bid}} C_u^{BS}(\alpha\phi)\right] = \mathrm{w}_{u,C^{Bid}} C_t^{BS}(\alpha\phi). \tag{A.28}$$

The second position consists of $\mathrm{w}_{u,C^{Bid}}$ units in the call option buyer's hedge portfolio that is short in the stock. This position has a value of $\mathrm{w}_{u,C^{Bid}}\left(-C_u^{BS}(\phi)\right)$ at time-u with its current value given by

$$V_t\left[\mathrm{w}_{u,C^{Bid}}\left(-C_u^{BS}(\phi)\right)\right] = \mathrm{w}_{u,C^{Bid}}\left(-C_t^{BS}(\phi)\right). \tag{A.29}$$

Summing (A.28) and (A.29) gives the current value of the random payoff (A.27) as

$$V_t\left[C_u^{BS}(\alpha\phi) - C_u^{Bid}\right] = \mathrm{w}_{u,C^{Bid}} C_t^{BS}(\alpha\phi) + \mathrm{w}_{u,C^{Bid}}\left(-C_t^{BS}(\phi)\right).$$

Substituting this into the call ask price representation (10) and rearranging gives

$$C_t^{Ask} = \left[1 - \int_t^T \mathrm{w}_{u,C^{Bid}} dF_{Cs}(u)\right] C_t^{BS}(\alpha\phi) + \left[\int_t^T \mathrm{w}_{u,C^{Bid}} dF_{Cs}(u)\right] C_t^{BS}(\phi). \tag{A.30}$$

We next match our conjectured forms (A.21)–(A.22) with the derived expressions in (A.26) and (A.30) and obtain the system for call weights as

$$\begin{aligned} \mathrm{w}_{t,C^{Bid}} &= 1 - \int_t^T \mathrm{w}_{u,C^{Ask}} dF_{Cb}(u) = 1 - \int_t^T \mathrm{w}_{u,C^{Ask}} \lambda_{Cb} e^{-\lambda_{Cb}(u-t)} du, && \text{(A.31)} \\ \mathrm{w}_{t,C^{Ask}} &= 1 - \int_t^T \mathrm{w}_{u,C^{Bid}} dF_{Cs}(u) = 1 - \int_t^T \mathrm{w}_{u,C^{Bid}} \lambda_{Cs} e^{-\lambda_{Cs}(u-t)} du. && \text{(A.32)} \end{aligned}$$

It is straightforward to check that the weights (17)–(18) in Proposition 2 solve the above system by substituting them into (A.31)–(A.32) and integrating simple exponential functions.[28] The deterministic nature of the derived weights verify that the call bid and ask prices indeed are as in (13)–(14) with the weights (17)–(18).

[28] Alternatively, these weights (17)–(18) can also be derived directly by differentiating the system (A.31)–(A.32) using the Leibniz integral rule, and solving the resulting system of two linear first-order differential equations simultaneously.

For the put bid and ask prices, we conjecture the forms

$$P_t^{Bid} = \left(1 - \mathrm{w}_{t,P^{Bid}}\right) P_t^{BS}(\phi) + \mathrm{w}_{t,P^{Bid}} P_t^{BS}(\alpha\phi), \tag{A.33}$$

$$P_t^{Ask} = \mathrm{w}_{t,P^{Ask}} P_t^{BS}(\phi) + \left(1 - \mathrm{w}_{t,P^{Ask}}\right) P_t^{BS}(\alpha\phi), \tag{A.34}$$

for all $t \leq T$ and the deterministic weight processes $\mathrm{w}_{t,P^{Bid}}$, $\mathrm{w}_{t,P^{Ask}}$. Going through the same steps as in the call option case yields the derived put bid and ask prices as

$$P_t^{Bid} = \left[\int_t^T \mathrm{w}_{u,P^{Ask}} dF_{Pb}(u)\right] P_t^{BS}(\phi) + \left[1 - \int_t^T \mathrm{w}_{u,P^{Ask}} dF_{Pb}(u)\right] P_t^{BS}(\alpha\phi), \tag{A.35}$$

$$P_t^{Ask} = \left[1 - \int_t^T \mathrm{w}_{u,P^{Bid}} dF_{Ps}(u)\right] P_t^{BS}(\phi) + \left[\int_t^T \mathrm{w}_{u,P^{Bid}} dF_{Ps}(u)\right] P_t^{BS}(\alpha\phi). \tag{A.36}$$

Matching the conjectured forms (A.33)–(A.34) with the derived expressions (A.35)–(A.36), we obtain the system for put weights as

$$\mathrm{w}_{t,P^{Bid}} = 1 - \int_t^T \mathrm{w}_{u,P^{Ask}} dF_{Pb}(u) = 1 - \int_t^T \mathrm{w}_{u,P^{Ask}} \lambda_{Pb} e^{-\lambda_{Pb}(u-t)} du,$$

$$\mathrm{w}_{t,P^{Ask}} = 1 - \int_t^T \mathrm{w}_{u,P^{Bid}} dF_{Ps}(u) = 1 - \int_t^T \mathrm{w}_{u,P^{Bid}} \lambda_{Ps} e^{-\lambda_{Ps}(u-t)} du.$$

Again, it is straightforward to check that the deterministic weights (19)–(20) in Proposition 2 solve the above system, verifying our conjecture. □

Proof of Proposition 3. Property (i) that the call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the shorting fee follows from the fact that these prices are weighted-averages of the no-arbitrage price bounds, which are both decreasing (call) and increasing (put) in the shorting fee (Proposition 1 properties (i)–(ii)), along with the fact that their weights do not depend on the shorting fee.

To prove property (ii) that both the call and put bid-ask spreads are increasing in the shorting fee for the given condition, we first obtain the bid-ask spread using (13)–(16) as

$$C_t^{Ask} - C_t^{Bid} = \left(\mathrm{w}_{t,C^{Ask}} + \mathrm{w}_{t,C^{Bid}} - 1\right) \left[C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)\right], \tag{A.37}$$

$$P_t^{Ask} - P_t^{Bid} = \left(\mathrm{w}_{t,P^{Ask}} + \mathrm{w}_{t,P^{Bid}} - 1\right) \left[P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)\right]. \tag{A.38}$$

Since the weights do not depend on the shorting fee ϕ, the bid-ask spreads are increasing in the shorting fee if and only if

$$\begin{aligned} \frac{\partial}{\partial \phi} C_t^{BS}(\alpha\phi) &> \frac{\partial}{\partial \phi} C_t^{BS}(\phi), \\ \frac{\partial}{\partial \phi} P_t^{BS}(\phi) &> \frac{\partial}{\partial \phi} P_t^{BS}(\alpha\phi). \end{aligned} \tag{A.39}$$

By using the partial derivative of the standard Black-Scholes call and put prices (A.9) and (A.11), we obtain these conditions as

$$\begin{aligned} \alpha e^{-\alpha\phi(T-t)} \Phi\left(d_1(\alpha\phi)\right) &< e^{-\phi(T-t)} \Phi\left(d_1(\phi)\right), \\ \alpha e^{-\alpha\phi(T-t)} \Phi\left(-d_1(\alpha\phi)\right) &< e^{-\phi(T-t)} \Phi\left(-d_1(\phi)\right). \end{aligned}$$

After rearranging the first condition gives the condition in property (ii) which is also a sufficient condition for the put since

$$\frac{\Phi\left(d_1(\phi)\right)}{\Phi\left(d_1(\alpha\phi)\right)} < \frac{\Phi\left(-d_1(\phi)\right)}{\Phi\left(-d_1(\alpha\phi)\right)}.$$

Property (iii) that the implied stock bid and ask prices are decreasing in the shorting fee follows immediately from differentiating their definitions (21)–(22) and employing the results in property (i) that the call bid and ask prices are decreasing, put bid and ask prices are increasing in the shorting fee, yielding

$$\begin{aligned} \frac{\partial}{\partial \phi} \tilde{S}_t^{Bid} &= \frac{\partial}{\partial \phi} C_t^{Bid} - \frac{\partial}{\partial \phi} P_t^{Ask} < 0, \\ \frac{\partial}{\partial \phi} \tilde{S}_t^{Ask} &= \frac{\partial}{\partial \phi} C_t^{Ask} - \frac{\partial}{\partial \phi} P_t^{Bid} < 0. \end{aligned}$$

□

Proof of Proposition 4. Property (i) that the call bid and ask prices are decreasing, while the put bid and ask prices are increasing in the partial lending follows from the fact that these prices are weighted-averages of the no-arbitrage price bounds which are either decreasing or do not depend on (call), and increasing or do not depend on (put) the partial lending (Proposition 1 properties (i)–(ii)), along with the fact that their weights do not depend on the partial lending.

Property (ii) that both the call and put bid-ask spreads are decreasing in the partial lending follows immediately from differentiating the call and put bid-ask spreads (A.37)–(A.38) with respect to the partial lending α. Since the weights do not depend on the partial lending, the bid-ask spreads are decreasing in the partial lending if and only if

$$\begin{aligned} \frac{\partial}{\partial \alpha} C_t^{BS}(\alpha \phi) &< \frac{\partial}{\partial \alpha} C_t^{BS}(\phi), \\ \frac{\partial}{\partial \alpha} P_t^{BS}(\phi) &< \frac{\partial}{\partial \alpha} P_t^{BS}(\alpha \phi), \end{aligned}$$

which always hold as (A.10) and (A.12) illustrate. □

Proof of Proposition 5. To determine the effects of the offsetting order arrival rates on option prices, we first derive their effects on the weights (17)–(20). The effects of the offsetting call sell and buy order arrival rates on the call weights are given by

$$\frac{\partial}{\partial \lambda_{Cs}} \mathrm{w}_{t,C^{Bid}} = \frac{\lambda_{Cb}}{(\lambda_{Cs}+\lambda_{Cb})^2} \left(1 - \left[1 + (\lambda_{Cs}+\lambda_{Cb})(T-t)\right] e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)}\right) > 0, \tag{A.40}$$

$$\frac{\partial}{\partial \lambda_{Cs}} \mathrm{w}_{t,C^{Ask}} = -\frac{\lambda_{Cb}}{(\lambda_{Cs}+\lambda_{Cb})^2} \left(1 - e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)}\right) - \frac{\lambda_{Cs}(T-t)}{\lambda_{Cs}+\lambda_{Cb}} e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)} < 0, \tag{A.41}$$

$$\frac{\partial}{\partial \lambda_{Cb}} \mathrm{w}_{t,C^{Bid}} = -\frac{\lambda_{Cs}}{(\lambda_{Cs}+\lambda_{Cb})^2} \left(1 - e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)}\right) - \frac{\lambda_{Cb}(T-t)}{\lambda_{Cs}+\lambda_{Cb}} e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)} < 0, \tag{A.42}$$

$$\frac{\partial}{\partial \lambda_{Cb}} \mathrm{w}_{t,C^{Ask}} = \frac{\lambda_{Cs}}{(\lambda_{Cs}+\lambda_{Cb})^2} \left(1 - \left[1 + (\lambda_{Cs}+\lambda_{Cb})(T-t)\right] e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)}\right) > 0, \tag{A.43}$$

where the signs of (A.40) and (A.43) follow from the fact that $1-(1+x)e^{-x} > 0$ for all $x > 0$. Similarly, the effects of the offsetting put buy and sell order arrival rates on the put weights are obtained immediately by substituting "put" for "call" in (A.40)–(A.43) as they have the same forms in (17)–(20), which yields

$$\frac{\partial}{\partial \lambda_{Ps}} \mathrm{w}_{t,P^{Bid}} > 0, \qquad \frac{\partial}{\partial \lambda_{Ps}} \mathrm{w}_{t,P^{Ask}} < 0, \qquad \frac{\partial}{\partial \lambda_{Pb}} \mathrm{w}_{t,P^{Bid}} < 0, \qquad \frac{\partial}{\partial \lambda_{Pb}} \mathrm{w}_{t,P^{Ask}} > 0. \tag{A.44}$$

Hence, property (i) that the call and put bid and ask prices are decreasing in their offsetting sell order arrival rates, while they are increasing in their offsetting buy order arrival rates follows by substituting (A.40)–(A.43) into

$$
\begin{aligned}
\frac{\partial}{\partial \lambda_{Cs}} C_t^{Bid} &= -\left[C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)\right] \frac{\partial}{\partial \lambda_{Cs}} \mathrm{w}_{t,C^{Bid}} < 0, \\
\frac{\partial}{\partial \lambda_{Cs}} C_t^{Ask} &= \left[C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)\right] \frac{\partial}{\partial \lambda_{Cs}} \mathrm{w}_{t,C^{Ask}} < 0, \\
\frac{\partial}{\partial \lambda_{Cb}} C_t^{Bid} &= -\left[C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)\right] \frac{\partial}{\partial \lambda_{Cb}} \mathrm{w}_{t,C^{Bid}} > 0, \\
\frac{\partial}{\partial \lambda_{Cb}} C_t^{Ask} &= \left[C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)\right] \frac{\partial}{\partial \lambda_{Cb}} \mathrm{w}_{t,C^{Ask}} > 0,
\end{aligned}
$$

and the respective inequalities in (A.44) into

$$
\begin{aligned}
\frac{\partial}{\partial \lambda_{Ps}} P_t^{Bid} &= -\left[P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)\right] \frac{\partial}{\partial \lambda_{Ps}} \mathrm{w}_{t,P^{Bid}} < 0, \\
\frac{\partial}{\partial \lambda_{Ps}} P_t^{Ask} &= \left[P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)\right] \frac{\partial}{\partial \lambda_{Ps}} \mathrm{w}_{t,P^{Ask}} < 0, \\
\frac{\partial}{\partial \lambda_{Pb}} P_t^{Bid} &= -\left[P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)\right] \frac{\partial}{\partial \lambda_{Pb}} \mathrm{w}_{t,P^{Bid}} > 0, \\
\frac{\partial}{\partial \lambda_{Pb}} P_t^{Ask} &= \left[P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)\right] \frac{\partial}{\partial \lambda_{Pb}} \mathrm{w}_{t,P^{Ask}} > 0.
\end{aligned}
$$

Property (ii) that both the call and put bid-ask spreads are decreasing in the offsetting order arrival rates follows immediately from differentiating the call and put bid-ask spreads (A.37)–(A.38) with respect to the arrival rates and obtain

$$
\begin{aligned}
\frac{\partial}{\partial \lambda_{Cs}} \left(C_t^{Ask} - C_t^{Bid}\right) &= -\left[C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)\right] (T-t)\, e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)} < 0, \\
\frac{\partial}{\partial \lambda_{Cb}} \left(C_t^{Ask} - C_t^{Bid}\right) &= -\left[C_t^{BS}(\alpha\phi) - C_t^{BS}(\phi)\right] (T-t)\, e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)} < 0, \\
\frac{\partial}{\partial \lambda_{Ps}} \left(P_t^{Ask} - P_t^{Bid}\right) &= -\left[P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)\right] (T-t)\, e^{-(\lambda_{Ps}+\lambda_{Pb})(T-t)} < 0, \\
\frac{\partial}{\partial \lambda_{Pb}} \left(P_t^{Ask} - P_t^{Bid}\right) &= -\left[P_t^{BS}(\phi) - P_t^{BS}(\alpha\phi)\right] (T-t)\, e^{-(\lambda_{Ps}+\lambda_{Pb})(T-t)} < 0.
\end{aligned}
$$

Property (iii) that the effects of the shorting fee on the call and put bid-ask spreads are decreasing in the offsetting order arrival rates follows immediately from differentiating the call bid-ask spreads (A.37)–(A.38) with respect to the arrival rates after substituting the fact

$$\mathrm{w}_{t,C^{Ask}} + \mathrm{w}_{t,C^{Bid}} - 1 = e^{-(\lambda_{Cs}+\lambda_{Cb})(T-t)},$$

and obtain

$$\begin{aligned}
\frac{\partial}{\partial \lambda_{Cs}} \frac{\partial}{\partial \phi} \left(C_t^{Ask} - C_t^{Bid} \right) &= -(T-t) \frac{\partial}{\partial \phi} \left(C_t^{Ask} - C_t^{Bid} \right), \\
\frac{\partial}{\partial \lambda_{Cb}} \frac{\partial}{\partial \phi} \left(C_t^{Ask} - C_t^{Bid} \right) &= -(T-t) \frac{\partial}{\partial \phi} \left(C_t^{Ask} - C_t^{Bid} \right), \\
\frac{\partial}{\partial \lambda_{Ps}} \frac{\partial}{\partial \phi} \left(P_t^{Ask} - P_t^{Bid} \right) &= -(T-t) \frac{\partial}{\partial \phi} \left(P_t^{Ask} - P_t^{Bid} \right), \\
\frac{\partial}{\partial \lambda_{Pb}} \frac{\partial}{\partial \phi} \left(P_t^{Ask} - P_t^{Bid} \right) &= -(T-t) \frac{\partial}{\partial \phi} \left(P_t^{Ask} - P_t^{Bid} \right).
\end{aligned}$$

Going through similar steps also gives the property that the effects of the partial lending on the call and put bid-ask spreads are decreasing in the offsetting order arrival rates. □

Proof of Proposition 6. Property (i) that the call bid and ask prices of banned stocks are lower, while the put bid and ask prices of banned stocks are higher than those of unbanned stocks follows by comparing the call bid and ask prices of banned stocks

$$C_{t,Ban}^{Bid} = \left(1 - \mathrm{w}_{t,C^{Bid}}\right) C_t^{BS}(\alpha\phi) + \mathrm{w}_{t,C^{Bid}} C_t^{BS}(2\phi), \tag{A.45}$$

$$C_{t,Ban}^{Ask} = \mathrm{w}_{t,C^{Ask}} C_t^{BS}(\alpha\phi) + \left(1 - \mathrm{w}_{t,C^{Ask}}\right) C_t^{BS}(2\phi), \tag{A.46}$$

and the put bid and ask prices of banned stocks

$$P_{t,Ban}^{Bid} = \left(1 - \mathrm{w}_{t,P^{Bid}}\right) P_t^{BS}(2\phi) + \mathrm{w}_{t,P^{Bid}} P_t^{BS}(\alpha\phi), \tag{A.47}$$

$$P_{t,Ban}^{Ask} = \mathrm{w}_{t,P^{Ask}} P_t^{BS}(2\phi) + \left(1 - \mathrm{w}_{t,P^{Ask}}\right) P_t^{BS}(\alpha\phi), \tag{A.48}$$

with the call and put bid and ask prices of unbanned stocks in (13)–(16) along with the facts that $C_t^{BS}(2\phi) < C_t^{BS}(\phi)$ and $P_t^{BS}(\phi) < P_t^{BS}(2\phi)$.

Property (ii) that both the call and put bid-ask spreads of banned stocks are higher than those of unbanned stocks follows immediately by comparing the option prices of banned stocks (A.45)–(A.48) with those of unbanned stocks (13)–(16) along with the facts that $C_t^{BS}(2\phi) < C_t^{BS}(\phi)$ and $P_t^{BS}(\phi) < P_t^{BS}(2\phi)$.

Property (iii) that the implied stock bid and ask prices of banned stocks are lower than those of unbanned stocks follows immediately from the definitions of implied stock prices (21)–(22) along with property (i) that the call bid and ask prices of banned stocks are lower, while the put bid and ask prices of banned stocks are higher than those of unbanned stocks.

Property (iv) that the call bid price decreases more than the ask price, while the put ask price increases more than the bid price of banned stocks follows by observing this property

$$\begin{aligned} C_{t,Ban}^{Bid} - C_t^{Bid} &< C_{t,Ban}^{Ask} - C_t^{Ask}, \\ P_{t,Ban}^{Bid} - P_t^{Bid} &< P_{t,Ban}^{Ask} - P_t^{Ask}, \end{aligned}$$

being equivalent to property (ii) that both the call and put bid-ask spreads of banned stocks are higher than those of unbanned stocks, after rearranging.

Appendix B: Additional Quantitative Analysis

In this Appendix, we provide additional tables to demonstrate further the quantitative effects of short-selling costs. In particular, Table 4 provides the upper and lower no-arbitrage bounds and their percentage deviations from the Black-Scholes prices for call and put options for the parameter values presented in Table 1. The last column gives the relative no-arbitrage range as it presents the ratio of the range to mid-points of the upper and lower bounds. Tables 5 and 6 provide the quantitative effects of costly short-selling as in Table 2, but for a shorter and a longer option maturity date of 1.5 and 4.5 months, respectively, while keeping all other parameter values as in Table 1.

Panel (a): Call option

Shorting fee decile	Option moneyness	BS price	Lower bound		Upper bound		No-arbitrage range	
	$\frac{K}{S_t}$	C_t^{BS}	$C_t(\phi)$		$C_t(\alpha\phi)$		$C_t(\alpha\phi) - C_t(\phi)$	
D1	1.10	1.43	1.43	-0.04%	1.43	0.00%	0.00	0.04%
D10		1.43	1.24	-13.45%	1.39	-2.94%	0.15	11.44%
D1	1.00	**2.63**	**2.63**	**-0.03%**	**2.63**	**0.00%**	**0.00**	**0.03%**
D10		**2.63**	**2.34**	**-11.22%**	**2.57**	**-2.43%**	**0.23**	**9.43%**
D1	0.90	4.46	4.46	-0.03%	4.46	0.00%	0.00	0.03%
D10		4.46	4.06	-9.06%	4.38	-1.94%	0.32	7.53%

Panel (b): Put option

	$\frac{K}{S_t}$	P_t^{BS}	$P_t(\alpha\phi)$		$P_t(\phi)$		$P_t(\phi) - P_t(\alpha\phi)$	
D1	1.10	4.49	4.49	0.00%	4.49	0.02%	0.00	0.02%
D10		4.49	4.56	1.68%	4.85	8.10%	0.29	6.12%
D1	1.00	**2.49**	**2.49**	**0.00%**	**2.49**	**0.03%**	**0.00**	**0.03%**
D10		**2.49**	**2.54**	**2.14%**	**2.75**	**10.46%**	**0.21**	**7.83%**
D1	0.90	1.11	1.11	0.00%	1.11	0.04%	0.00	0.04%
D10		1.11	1.14	2.75%	1.26	13.60%	0.12	10.03%

Table 4: **No-arbitrage bounds for option prices.** This table reports the no-arbitrage upper and lower bounds and their percentage deviations from the Black-Scholes prices for a call option (Panel (a)) and a put option (Panel (b)) on a typical stock in the lowest (D1) and the highest (D10) shorting fee decile in Drechsler and Drechsler (2016) for three different option moneyness levels. The last column gives the ratio of the no-arbitrage range to mid-points of the upper and lower bounds. All parameter values are as in Table 1.

Panel (a): Call option

Shorting fee decile	Option moneyness	BS price	Bid price	Ask price	Relative change	Bid-Ask spread	Relative spread	Implied volatility
	$\frac{K}{S_t}$	C_t^{BS}	C_t^{Bid}	C_t^{Ask}	$\frac{C_t^{Mid}-C_t^{BS}}{C_t^{BS}}$	$C_t^{Ask}-C_t^{Bid}$	$\frac{C_t^{Ask}-C_t^{Bid}}{C_t^{Mid}}$	$\tilde{\sigma}_{t,C^{Mid}}$
D1	1.10	0.73	0.73	0.73	-0.02%	0.00	0.01%	40.00%
D10		0.73	0.68	0.69	-6.23%	0.01	2.14%	38.80%
D1	1.00	**1.85**	**1.85**	**1.85**	**-0.01%**	**0.00**	**0.01%**	**40.00%**
D10		**1.85**	**1.75**	**1.78**	**-4.78%**	**0.03**	**1.63%**	**38.05%**
D1	0.90	3.84	3.84	3.84	-0.01%	0.00	0.00%	39.99%
D10		3.84	3.68	3.73	-3.47%	0.04	1.17%	35.71%

Panel (b): Put option

	$\frac{K}{S_t}$	P_t^{BS}	P_t^{Bid}	P_t^{Ask}	$\frac{P_t^{Mid}-P_t^{BS}}{P_t^{BS}}$	$P_t^{Ask}-P_t^{Bid}$	$\frac{P_t^{Ask}-P_t^{Bid}}{P_t^{Mid}}$	$\tilde{\sigma}_{t,P^{Mid}}$
D1	1.10	3.87	3.87	3.87	0.01%	0.00	0.00%	40.01%
D10		3.87	3.97	4.01	3.18%	0.04	1.01%	43.17%
D1	1.00	**1.78**	**1.78**	**1.78**	**0.01%**	**0.00**	**0.01%**	**40.00%**
D10		**1.78**	**1.84**	**1.87**	**4.52%**	**0.03**	**1.42%**	**41.78%**
D1	0.90	0.55	0.55	0.55	0.02%	0.00	0.01%	40.00%
D10		0.55	0.58	0.60	6.40%	0.01	1.99%	41.10%

Table 5: **Quantitative effects of costly short-selling for shorter maturity options.** This table reports the effects of costly short-selling for a shorter maturity (1.5 months) call option (Panel (a)) and a put option (Panel (b)) on a typical stock in the lowest (D1) and the highest (D10) shorting fee decile in Drechsler and Drechsler (2016) for three different option moneyness levels. C_t^{Mid} and P_t^{Mid} denote the mid-point prices of the call and put, e.g., $C_t^{Mid} = 0.5(C_t^{Ask} + C_t^{Bid})$ and $P_t^{Mid} = 0.5(P_t^{Ask} + P_t^{Bid})$. Implied volatilities in the last columns are obtained by employing the standard approach of inverting the Black-Scholes formula using the mid-point option prices as inputs. All other parameter values are as in Table 1.

Panel (a): Call option

Shorting fee decile	Option moneyness	BS price	Bid price	Ask price	Relative change	Bid-Ask spread	Relative spread	Implied volatility
	$\frac{K}{S_t}$	C_t^{BS}	C_t^{Bid}	C_t^{Ask}	$\frac{C_t^{Mid}-C_t^{BS}}{C_t^{BS}}$	$C_t^{Ask}-C_t^{Bid}$	$\frac{C_t^{Ask}-C_t^{Bid}}{C_t^{Mid}}$	$\tilde{\sigma}_{t,C^{Mid}}$
D1	1.10	2.00	2.00	2.00	-0.03%	0.00	0.01%	39.99%
D10		2.00	1.78	1.84	-9.73%	0.06	3.44%	37.44%
D1	1.00	**3.24**	**3.24**	**3.24**	**-0.02%**	**0.00**	**0.01%**	**39.99%**
D10		**3.24**	**2.92**	**3.01**	**-8.41%**	**0.09**	**2.94%**	**36.50%**
D1	0.90	4.98	4.98	4.98	-0.02%	0.00	0.01%	39.99%
D10		4.98	4.57	4.69	-7.09%	0.11	2.46%	34.60%

Panel (b): Put option

	$\frac{K}{S_t}$	P_t^{BS}	P_t^{Bid}	P_t^{Ask}	$\frac{P_t^{Mid}-P_t^{BS}}{P_t^{BS}}$	$P_t^{Ask}-P_t^{Bid}$	$\frac{P_t^{Ask}-P_t^{Bid}}{P_t^{Mid}}$	$\tilde{\sigma}_{t,P^{Mid}}$
D1	1.10	4.99	4.99	4.99	0.02%	0.00	0.01%	40.01%
D10		4.99	5.24	5.34	6.17%	0.10	1.91%	44.00%
D1	1.00	**3.02**	**3.02**	**3.02**	**0.02%**	**0.00**	**0.01%**	**40.01%**
D10		**3.02**	**3.21**	**3.29**	**7.63%**	**0.08**	**2.34%**	**42.96%**
D1	0.90	1.57	1.57	1.57	0.02%	0.00	0.01%	40.01%
D10		1.57	1.69	1.74	9.50%	0.05	2.88%	42.23%

Table 6: **Quantitative effects of costly short-selling for longer maturity options.** This table reports the effects of costly short-selling for a longer maturity (4.5 months) call option (Panel (a)) and a put option (Panel (b)) on a typical stock in the lowest (D1) and the highest (D10) shorting fee decile in Drechsler and Drechsler (2016) for three different option moneyness levels. C_t^{Mid} and P_t^{Mid} denote the mid-point prices of the call and put, e.g., $C_t^{Mid} = 0.5(C_t^{Ask} + C_t^{Bid})$ and $P_t^{Mid} = 0.5(P_t^{Ask} + P_t^{Bid})$. Implied volatilities in the last columns are obtained by employing the standard approach of inverting the Black-Scholes formula using the mid-point option prices as inputs. All other parameter values are as in Table 1.

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