Endogenous Group Formation in Contests: Unobservable Sharing Rules

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Abstract
We study contests in which the winner shares the prize with the other players in his group, if any, and each group's sharing rule is unobservable to the players in the other groups and the singletons when they expend their effort. First, in the basic model, the number of groups, their sizes, and the number of singletons are exogenously given. We compare the outcomes of the case of unobservable sharing rules with those of the case of observable sharing rules. Next, in the main model, the number of groups, their sizes, and the number of singletons are endogenously determined. We obtain the equilibrium numbers of groups, the equilibrium group sizes, and the equilibrium numbers of singletons; and examine the effect of endogenous group formation on total effort level and the profitability of endogenous group formation. Overall, we obtain quite different results from those of the case of observable sharing rules.

JEL classification: D72, D74

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1. Introduction

Contests are common in which players, each as a member of a group or as a singleton (or, equivalently, a nonmember), compete with one another by expending irreversible effort or resources to win a prize. Examples include various types of rent-seeking contests, patent competition among consortiums, competition among political parties or candidates for office, class action litigation, and sporting contests. In some contests, group formation may occur because players want to share the risk of failure with others. In others, group formation may occur because players try to gain strategic advantage through group formation.

In such contests, the players in each group jointly decide how to share the prize among themselves if they or one of them wins it—that is, they make a binding agreement on their sharing rule. One example for such an agreement comes from college football, where the winning team of a bowl game shares or divides the prize with the rest of the teams in its conference according to the sharing rule of the conference. Another example comes from competition among university professors and researchers for research grants. If a researcher wins a grant, all of the researchers in a university may share the winning grant in the form of overhead or indirect costs collected by the university.

Here we may well expect that each group's sharing rule is unobservable to the players in the other groups and the singletons when they expend their effort. For example, in competition among political parties for (presidential) office, members of one political party may not be sure, when they campaign, how members of another party will share office benefits, policy influence, and the political posts among themselves in case of winning. This situation of unobservable sharing rules may occur simply because the groups do not announce their sharing rules. Furthermore, it may occur, even though they announce publicly their sharing rules, because their announced sharing rules are unverifiable. Nitzan and Ueda (2011) say that, without restrictive assumptions that decisions made within a group are transparent and detection of changes is easy, a model of group contests with observable sharing rules is questionable.
In such contests, we may well expect also that the number of groups, their sizes, and the number of singletons are endogenously determined. It is these ideas, unobservable sharing rules and endogenous group formation, that motivate this paper.

Accordingly, this paper studies contests in which the winner shares the prize with the other players in his group, if any, and each group's sharing rule is unobservable to the players in the other groups and the singletons when they expend their effort. It considers two models: the basic model and the main model. In both models, the prize is awarded to one of the players.

In the basic model, the number of groups, their sizes, and the number of singletons are exogenously given. We formally consider the following game. First, the players in each group jointly decide how to share the prize among themselves if one of them wins it — that is, they make a binding agreement on their sharing rule (or, equivalently, their winner's fractional share). Next, all the players in the contest choose their effort levels simultaneously and independently. When the players choose their effort levels, each group's sharing rule is unobservable to the players in the other groups and the singletons. Finally, the winning player is determined, and the players in his group, if any, share the prize according to their sharing rule on which they agreed.

We show that each group's equilibrium effort level and the equilibrium total effort level are independent of the number of players in the contest and the sizes of the groups, as long as changes in these do not change the number of groups or the number of singletons. We show also that each player in a larger group expends less effort, and has a less equilibrium expected payoff, than each player in a smaller one. Finally, we show that the equilibrium expected payoff for each singleton is greater than that for each player in any actual group.

We compare the outcomes of the case of unobservable sharing rules — the game presented above — with those of the case of observable sharing rules, the game which is the same as the one presented above with the exception that each group's sharing rule is now observable to all the players in the game. By the situation of "observable sharing rules," we mean a situation in which the groups announce publicly their sharing rules and, furthermore, are committed to their sharing rules. We show that the two cases yield the same outcomes if there are one group
and one singleton, whereas they yield different outcomes if there is more than one singleton or more than one group.

Next, we consider the main model. The main model is the same as the basic model with the exception that, in the main model, the players decide first whether to form groups. Thus, in the main model, the number of groups, their sizes, and the number of singletons are endogenously determined. We first obtain the equilibrium numbers of groups, the equilibrium group sizes, and the equilibrium numbers of singletons. Then, after obtaining the equilibrium total effort level and each player's equilibrium expected payoff, we examine the effect of endogenous group formation on total effort level and the profitability of endogenous group formation. We show the following. If the number of players in the contest is four or smaller, then group formation occurs. If the number of players is five or greater, then the individual contest occurs – that is, group formation does not occur. For any number of players, the equilibrium total effort level does not exceed the total effort level resulting from the individual contest – in other words, endogenous group formation does not increase prize dissipation, as compared with the individual contest. If the number of players is four or smaller, then endogenous group formation is beneficial to the players in the contest, as compared with the individual contest. Overall, we obtain quite different results from those in the corresponding model with observable sharing rules.

This paper is related to the literature on collective rent seeking, and to the literature on endogenous group formation in contests or conflicts. In these literatures, a standard assumption is that, when players choose their effort levels, each group's sharing rule, if any, is observable to all the players – in other words, public information is assumed regarding sharing rules. Nitzan (1991a, 1991b), Baik (1994), Baik and Shogren (1995), Hausken (1995), Lee (1995), Davis and Reilly (1999), Ueda (2002), Baik et al. (2006), and Ursprung (2012) are examples of collective rent seeking with observable sharing rules. Baik and Lee (2001) consider collective rent seeking with observable sharing rules in which the number of groups, their sizes, and the number of singletons are endogenously determined, and examine the profitability of endogenous group
formation and the effect of endogenous group formation on rent dissipation. Garfinkel (2004) considers a three-stage model of distributional conflict with observable sharing rules in which individuals can form alliances in the first stage, and examine the effect of endogenous alliance formation on the severity of conflict. Bloch et al. (2006) extend the model of conflicts and contests introduced by Esteban and Ray (1999) to incorporate endogenous group formation. Sánchez-Páges (2007) considers endogenous coalition formation in contests. Bloch (2012) provides a survey of the theoretical literature on alliance formation, and then considers different models of endogenous formation of alliances in conflicts. Recently, unlike the previous papers, Baik and Lee (2007) and Nitzan and Ueda (2011) consider collective rent seeking between groups in which each group's sharing rule is private information. Baik and Lee (2012) study collective rent seeking between two groups in which each group has the option of making its sharing rule observable or unobservable. They show that the case where both groups make their sharing rules observable does not occur in equilibrium. A striking difference between the current paper and the previous papers in the above-mentioned literatures is that, unlike the previous papers, the current paper studies contests with both unobservable sharing rules and endogenous group formation.

This paper is also related to the literature on the theory of endogenous coalition formation: See, for example, Bloch (1995), Yi (1998), Belleflamme (2000), Morasch (2000), and Yi and Shin (2000). These papers concentrate on examining the equilibrium structures of coalitions – specifically, associations, R&D joint ventures, or strategic alliances of firms – in oligopolies.

The paper proceeds as follows. Section 2 develops the basic model, and sets up the game in which the number of groups, their sizes, and the number of singletons are exogenously given. In Section 3, we solve the game, and obtain the equilibrium sharing rules of the groups, the equilibrium effort levels of the players, and the equilibrium expected payoffs for the players. In Section 4, we compare the outcomes of the case of unobservable sharing rules with those of the case of observable sharing rules. In Section 5, we consider the main model in which the number
of groups, their sizes, and the number of singletons are endogenously determined. Finally, Section 6 offers our conclusions, and discusses modifications and extensions of the models.

2. The basic model: \( N \) exogenously given groups

There are \( n \) risk-neutral players who compete by expending irreversible effort to win a rent (or, in general, a prize), where \( n \geq 3 \). They each are a member of one of \( N \) groups, 1 through \( N \), or do not belong to any of these groups. The prize will be awarded to one of the players. Every player values the prize at \( V \). We assume that the players' valuation \( V \) for the prize is positive and publicly known.

Group \( i \), for \( i = 1, \ldots, N \), consists of \( m_i \) players, where \( N \geq 1 \) and \( 2 \leq m_i < n \). Without loss of generality, we assume that \( m_1 \geq m_2 \geq \ldots \geq m_N \). The players in group \( i \) write an agreement, before they choose their effort levels, on how to share or divide the prize if one of them wins the prize. The agreement takes the following form: If a player in group \( i \) wins the prize, then the winner takes \( \sigma_i V \) and each of the other players in the group takes \( (1 - \sigma_i)V/(m_i - 1) \), where \( \sigma_i \geq 1/m_i \). We call \( \sigma_i \) group \( i \)'s winner's fractional share or, broadly, its sharing rule. If \( 1/m_i \leq \sigma_i < 1 \), then the winner shares the prize with the other players in his group, and thus he takes less than the prize. If \( \sigma_i = 1 \), then the winner takes all the prize and nothing is left for the other players in his group. If \( \sigma_i > 1 \), then the winner takes "incentives" from the other players in his group as well as the prize.

Note that a group as defined in the preceding paragraph is an extremely loose form of association, and may be like a network of solidarity: Even the players in the same group compete with each other to win the prize. Section 6 will discuss a model in which the players in each group pool their effort to win the prize and the prize is awarded to one of the singletons or one of the groups. Note also that the players in each group create externalities for each other when they expend their effort.

We may treat the singletons — that is, the players who do not belong to any of those \( N \) groups — as the members of group \( N+1 \) whose winner's fractional share is unity and publicly
known. Let \( m_{N+1} \) denote the size of group \( N+1 \) or, equivalently, the number of singletons, where \( m_{N+1} \geq 0 \). We have then \( n = \sum_{i=1}^{N+1} m_i \).

Let \( x_{ik} \) represent the effort level that player \( k \) in group \( i \) expends, for \( i = 1, \ldots, N+1 \), and let \( X_i \) represent the effort level that all the players in group \( i \) expend, so that \( X_i = \sum_{k=1}^{m_i} x_{ik} \). Let \( X \) represent the effort level that all the players in the contest expend, so that \( X = \sum_{i=1}^{N+1} X_i \). Each player's effort level is nonnegative, and measured in units commensurate with the prize. Let \( p_{ik} \) denote the probability that player \( k \) in group \( i \) wins the prize. We assume that the contest success function for player \( k \) in group \( i \) is

\[
p_{ik} = \begin{cases} 
  x_{ik}/X & \text{for } X > 0 \\
  1/n & \text{for } X = 0.
\end{cases}
\]

Let \( \pi_{ik} \) denote the expected payoff for player \( k \) in group \( i \). Then the payoff function for player \( k \) in group \( i \) is

\[
\pi_{ik} = (\sigma_i V - x_{ik})p_{ik} + \{(1 - \sigma_i)V/(m_i - 1) - x_{ik}\} \sum_{j \neq k} p_{ij} + (-x_{ik})(1 - \sum_{j=1}^{m_i} p_{ij})
\]

\[
= \sigma_i V p_{ik} + \{(1 - \sigma_i)V/(m_i - 1)\} \sum_{j \neq k} p_{ij} - x_{ik} \quad \text{for } i = 1, \ldots, N,
\]

and

\[
\pi_{ik} = V p_{ik} - x_{ik} \quad \text{for } i = N+1,
\]

where \( \sum_{j \neq k} p_{ij} \) is the probability that any one of the other players in group \( i \) wins the prize, and \( (1 - \sum_{j=1}^{m_i} p_{ij}) \) is the probability that no player in group \( i \) wins it.

We formally consider the following game. First, the players in group \( i \), for \( i = 1, \ldots, N \), jointly decide how to share the prize among themselves if one of them wins it. That is, they make a binding agreement on their sharing rule \( \sigma_i \). Note that, because the players in the group are identical, their decision on \( \sigma_i \) is unanimous. Next, all the players in the contest choose their
effort levels simultaneously and independently. When the players choose their effort levels, the players in each group know the sharing rule of their own group; however, each group's sharing rule is unobservable to the players in the other groups and the singletons. Lastly, the winning player is determined, and the players in his group, if any, share the prize according to their sharing rule on which they agreed.

This game differs from a standard two-stage game in which the actions chosen in the first stage are observed by all the players before actions are chosen in the second stage. In the current game, the players in group $i$, for $i = 1, \ldots, N$, choose two sequential actions — specifically, their sharing rule and then effort levels — without observing those chosen by the players in the other groups or the effort levels chosen by the singletons. Thus they play a simultaneous-move game with the players in the other groups and the singletons.6

Finally, we assume that all of the above is common knowledge among the players.

3. Equilibrium sharing rules, effort levels, and expected payoffs

The equilibrium sharing rules of groups 1 through $N$ and the equilibrium effort levels of the players in the contest satisfy the following two requirements. First, each player's effort level is optimal given his group's sharing rule and given the effort levels of all the other players. In other words, each player's effort level is a best response both to his group's sharing rule and to the effort levels of the other players. Second, group $i$'s sharing rule, for $i = 1, \ldots, N$, is optimal given the effort levels of the players in the other groups and the singletons and given the subsequent behavior, or rather effort levels, of its own players.7

To obtain the equilibrium sharing rules of the groups and the equilibrium effort levels of the players, working backward, we consider first the players' decisions on their effort levels, and then consider the players' decisions on their groups' sharing rules.

Consider the decision of player $k$ in group $i$, for $i = 1, \ldots, N+1$, on his effort level. After observing only his group's sharing rule $\sigma_i$, he seeks to maximize his expected payoff (2) over his
effort level \( x_{ik} \), taking the effort levels of all the other players as given. From the first-order condition for maximizing function (2), we obtain
\[
\sigma_i V(X - x_{ik}) - (1 - \sigma_i) V(X_i - x_{ik})/(m_i - 1) = X^2 
\]
(3)
or, equivalently,
\[
x_{ik} = \sqrt{\sigma_i V(X - x_{ik}) - (1 - \sigma_i) V(X_i - x_{ik})/(m_i - 1)} - (X - x_{ik}).
\]

It is straightforward to see that \( \pi_{ik} \) in (2) is strictly concave in \( x_{ik} \), and thus the second-order condition for maximizing function (2) is satisfied. Because the players are identical within each group, they choose the same effort level in equilibrium. Thus, let \( x_{ik} = x_i \) for all \( k \). Then equation (3) reduces to
\[
\sigma_i Q - V x_i = Q^2,
\]
where \( Q = \sum_{i=1}^{N+1} m_i x_i \). Using this equation, we obtain
\[
x_i(\sigma_i, Q_{-i}) = \{(\sigma_i m_i - 1) V - 2m_iQ_{-i} + \sqrt{(\sigma_i m_i - 1)^2 V^2 + 4m_iVQ_{-i}}\}/2m_i^2, \quad (4)
\]
where \( Q_{-i} = m_1x_1 + \cdots + m_{i-1}x_{i-1} + m_{i+1}x_{i+1} + \cdots + m_{N+1}x_{N+1} \).

Next, consider the decision of the players in group \( i \), for \( i = 1, \ldots, N \), on their group's sharing rule. Recall that \( \sigma_{N+1} = 1 \), which is known publicly. Taking \( Q_{-i} \) as given, the players in group \( i \) seek to maximize their expected payoffs over their sharing rule \( \sigma_i \), having perfect foresight about \( x_i(\sigma_i, Q_{-i}) \) for each value of \( \sigma_i \). More precisely, they seek to maximize
\[
\pi_i(\sigma_i, Q_{-i}) = V x_i(\sigma_i, Q_{-i})/\{m_i x_i(\sigma_i, Q_{-i}) + Q_{-i}\} - x_i(\sigma_i, Q_{-i}) \quad (5)
\]
with respect to \( \sigma_i \), taking \( Q_{-i} \) as given. Note that we obtain function (5) using functions (1), (2), and (4). From the first-order condition for maximizing function (5), we obtain
\[
\sigma_i(Q_{-i}) = \{1 + (m_i - 1)\sqrt{Q_{-i}/V}\}/m_i. \quad (6)
\]
We are now ready to obtain the equilibrium sharing rules of the $N$ groups and the equilibrium effort levels of the players. They satisfy $N+1$ equations from (4) and $N$ equations from (6) simultaneously. Let $\sigma_i(m)$ represent the equilibrium sharing rule of group $i$, for $i = 1, \ldots, N$, and let $x_i(m)$ represent the equilibrium effort level of each player in group $i$, for $i = 1, \ldots, N+1$, where $m = (m_1, \ldots, m_{N+1})$. Substituting function (6) into function (4), we have

$$x_i(Q_{-i}) = (\sqrt{VQ_{-i} - Q_{-i}})/m_i$$

(7)

or, equivalently,

$$Q_{-i}^2 = VQ_{-i}$$

for $i = 1, \ldots, N$.  

(8)

Substituting $\sigma_{N+1} = 1$ into function (4), we have

$$Q^2 = V(Q - x_{N+1}).$$

(9)

By solving $N+1$ simultaneous equations from (8) and (9), we obtain the players' equilibrium effort levels, $x_1(m)$ through $x_{N+1}(m)$ (see Appendix A). Next, substituting the players' equilibrium effort levels into function (6), we obtain the groups' equilibrium sharing rules, $\sigma_1(m)$ through $\sigma_N(m)$. Lemma 1 reports the groups' equilibrium sharing rules, and Lemma 2 reports the effort levels of the players, those of the groups, and the total effort level in equilibrium.

**Lemma 1.** Group $i$'s equilibrium sharing rule is $\sigma_i(m) = 1 - (m_i - 1)/m_i(N + m_{N+1})$ for $i = 1, \ldots, N$.

It is immediate from Lemma 1 that group $i$'s equilibrium winner's fractional share is greater than $1/m_i$, and is less than unity. Effort expended by a player generates a negative externality for the other players in his group since it reduces the other players' winning probabilities. We may say that each group's equilibrium winner's fractional share, since it is less than unity, works as a device to amend this negative externality. Another result from Lemma 1 is that each group's equilibrium sharing rule is independent of the players' valuation $V$ for the
prize. This can be explained as follows. If $V$ increases, for example, the players in each group do not decrease or increase their winner's fractional share because the higher valuation serves equally to motivate more the other players as well as the players in the group. We obtain from Lemma 1 that, as $m_i$ increases without changing $N$ or changing $m_{N+1}$, group $i$'s equilibrium winner's fractional share decreases: $\partial \sigma_i(m) / \partial m_i < 0$. This makes intuitive sense. Competing with the players in the other groups and the singletons to win the prize, the players in group $i$ optimally motivate themselves to win the prize by choosing the "right" sharing rule. If $m_i$ increases, they can do so with a lower value of their winner's fractional share because their larger group size, too, serves to attain their goal. Note that an increase in $m_i$, without changing $N$ or changing $m_{N+1}$, may occur if $n$ increases or $m_j$ decreases for some $j$, where $j = 1, \ldots, N$ with $i \neq j$. We also obtain from Lemma 1 that group $i$'s equilibrium winner's fractional share increases as $N$ increases without changing $m_{N+1}$ or changing $m_i$; it increases as $m_{N+1}$ increases without changing $N$ or changing $m_i$; and it increases as both $N$ and $m_{N+1}$ increase without changing $m_i$. An increase in $N$ or $m_{N+1}$ increases the intensity of the competition, so that the players in group $i$ need to increase their winner's fractional share. Finally, according to Lemma 1, two groups of the same size have the same equilibrium sharing rule, and the equilibrium winner's fractional share of a larger group is less than that of a smaller one; consequently, we have $\sigma_1(m) \leq \ldots \leq \sigma_N(m)$.11

Let $Q_i(m)$ represent group $i$'s equilibrium effort level, and $Q(m)$ the equilibrium total effort level: $Q_i(m) = m_i x_i(m)$ and $Q(m) = \sum_{i=1}^{N+1} Q_i(m) = \sum_{i=1}^{N+1} m_i x_i(m)$.

**Lemma 2.** The equilibrium effort levels of the individual players, those of the groups, and the equilibrium total effort level are $x_i(m) = V(N + m_{N+1} - 1)/m_i(N + m_{N+1})^2$ for $i = 1, \ldots, N$; $x_{N+1}(m) = V(N + m_{N+1} - 1)/(N + m_{N+1})^2$; $Q_i(m) = V(N + m_{N+1} - 1)/(N + m_{N+1})^2$ for $i = 1, \ldots, N$; $Q_{N+1}(m) = Vm_{N+1}(N + m_{N+1} - 1)/(N + m_{N+1})^2$; and $Q(m) = V(N + m_{N+1} - 1)/(N + m_{N+1})$. 
Note first that all the players in the contest are active in equilibrium — that is, the players' equilibrium effort levels, $x_1(m)$ through $x_{N+1}(m)$, are positive. Lemma 2 says that group $i$'s equilibrium effort level, $Q_i(m)$, for $i = 1, \ldots, N+1$, and the equilibrium total effort level, $Q(m)$, depend on the number $N$ of groups and the number $m_{N+1}$ of singletons; however, they are independent of the number $n$ of players in the contest and the sizes of groups 1 through $N$, as long as changes in these do not change $N$ or $m_{N+1}$. This implies that, as $n$ increases by increasing only the sizes of one or more existing actual groups, neither the equilibrium effort levels of groups 1 through $N+1$ nor the equilibrium total effort level changes. An interesting and important observation from Lemma 2 is that group $i$'s equilibrium effort level, $Q_i(m)$, for $i = 1, \ldots, N$, and each singleton's equilibrium effort level, $x_{N+1}(m)$, are equal to the corresponding players' equilibrium effort levels obtained in a reduced contest in which only $(N + m_{N+1})$ players — one for each group, and $m_{N+1}$ singletons — compete individually to win the prize. We can understand this observation by noting that, in equation (7), group $i$'s effort level, $m_i x_i(Q_{-i})$, for $i = 1, \ldots, N$, does not depend on its own size.

Using Lemma 2, we obtain the following further results on the equilibrium effort levels. First, the equilibrium total effort level, $Q(m)$, is less than the players' valuation $V$ for the prize. Second, we have $Q_1(m) = \ldots = Q_N(m)$, and $Q_1(m) = x_{N+1}(m)$ if there is at least one singleton. That is, the equilibrium effort levels of the groups are the same regardless of the groups' different sizes. This happens because, given their different sizes, the groups choose their sharing rules which countervail the differences in group size — indeed, a smaller group chooses a larger winner's fractional share than a larger group in equilibrium. Third, each player in two groups of the same size expends the same effort, and each player in a larger group expends less effort than each player in a smaller one; consequently, we have $x_1(m) \leq \ldots \leq x_N(m)$. This result is natural because the groups' equilibrium effort levels are the same, and $m_1 \geq m_2 \geq \ldots \geq m_N$. Furthermore, it makes intuitive sense because a larger group, possessing a competitive size advantage over a smaller one, allows its players to ease up by choosing a lower winner's fractional share.
We end this section by looking at the expected payoffs for the players in equilibrium, which are used in studying endogenous group formation in Section 5. Let \( \pi_i(m) \) represent the equilibrium expected payoff for each player in group \( i \), for \( i = 1, \ldots, N+1 \). Using functions (1) and (2), and Lemmas 1 and 2, we obtain Lemma 3.

**Lemma 3.** Given the players' valuation \( V \) for the prize, we have

\[
\pi_i(m) = \frac{V}{m_i(N + m_{N+1})^2} \quad \text{for} \quad i = 1, \ldots, N, \quad \text{and} \quad \pi_{N+1}(m) = \frac{V}{(N + m_{N+1})^2}.
\]

Lemma 3 says that the equilibrium expected payoff for each player in group \( i \), for \( i = 1, \ldots, N+1 \), does not depend on the size \( m_j \) of group \( j \), for \( j = 1, \ldots, N \) with \( i \neq j \), as long as a change in \( m_j \) does not change \( m_i, N, \) or \( m_{N+1} \). Lemma 3 implies that each player in two groups of the same size has the same equilibrium expected payoff, and each player in a larger group has a less equilibrium expected payoff than each player in a smaller one; consequently, we have

\[
\pi_1(m) \leq \ldots \leq \pi_N(m).^{16}
\]

It implies also that \( m_1 \pi_1(m) = \ldots = m_N \pi_N(m) \), and that \( m_1 \pi_1(m) = \pi_{N+1}(m) \) if there is at least one singleton.\(^{17}\)

Another interesting result in Lemma 3 is that the equilibrium expected payoff for each singleton is greater than that for each player in any actual group. This makes intuitive sense because the singletons are more motivated than the players in any group — indeed, \( \sigma_{N+1} \) is greater than the equilibrium winner's fractional share of any group.

**4. Comparison with the case of observable sharing rules**

In this section, we compare the outcomes of the case of unobservable sharing rules with those of the case of observable sharing rules. The outcomes of the case of unobservable sharing rules are those of the game analyzed so far — in which each group's sharing rule is unobservable to the players in the other groups and the singletons — and are provided in Lemmas 1 through 3 in Section 3. The outcomes of the case of observable sharing rules are provided in Lemmas 2 through 4 in Baik and Lee (2001). By the case of observable sharing rules, we mean the game
which is the same as the one in Section 2 with the exception that each group's sharing rule is now observable to all the players in the game. More specifically, we refer to the following two-stage game as the case of observable sharing rules. In the first stage, the players in group $i$, for $i = 1, \ldots, N$, jointly make a binding agreement on their sharing rule $\sigma_i$, and then all the groups simultaneously announce (and commit to) their sharing rules. In the second stage, after knowing the sharing rules of the $N$ groups, all the players in the contest choose their effort levels simultaneously and independently. The winning player is determined at the end of the second stage, and the players in his group, if any, share the prize according to their sharing rule announced in the first stage.

Table 1 summarizes the outcomes of the case of unobservable sharing rules and those of the case of observable sharing rules. Comparing the outcomes of the two cases, we obtain Proposition 1. The superscripts $ub$ and $ob$ in Proposition 1 indicate the outcomes of the case of unobservable sharing rules and those of the case of observable sharing rules, respectively.

**Proposition 1.** (a) If there are one group and one singleton, then the two cases yield the same outcomes. (b) If there are one group and more than one singleton, then we obtain: (i) $\sigma_i^{ub} < \sigma_i^{ob}$, (ii) $Q_i^{ub} < Q_i^{ob}$, (iii) $Q_{N+1}^{ub} > Q_{N+1}^{ob}$, (iv) $Q^{ub} < Q^{ob}$, (v) $\pi_i^{ub} < \pi_i^{ob}$, and (vi) $\pi_{N+1}^{ub} > \pi_{N+1}^{ob}$, for $i = 1, \ldots, N$. (c) If there is more than one group, then we obtain: (i) $\sigma_i^{ub} < \sigma_i^{ob}$, (ii) $Q_i^{ub} < Q_i^{ob}$ unless $n$ is large, (iii) $Q_{N+1}^{ub} > Q_{N+1}^{ob}$, (iv) $Q^{ub} < Q^{ob}$, (v) $\pi_i^{ub} > \pi_i^{ob}$ unless $n$ is sufficiently large, and (vi) $\pi_{N+1}^{ub} > \pi_{N+1}^{ob}$, for $i = 1, \ldots, N$.

Part (a) of Proposition 1 happens because the players in the group choose the same sharing rule in the two cases. Facing just one singleton, the players in the group do not behave differently between the two cases.

On the other hand, parts (b) and (c) say that the unobservability of sharing rules makes differences, as compared to the case of observable sharing rules, if there is more than one singleton or more than one group – that is, if the competition is fiercer. Most importantly, group
i's equilibrium winner's fractional share is less in the case of unobservable sharing rules than in the case of observable sharing rules, which provides the intuitions behind the rest of the comparative results. This can be explained as follows. In the case of observable sharing rules, the players in a group can make a strategic commitment for the competition in the effort-expending stage by announcing, in the first stage, their sharing rule. Naturally, to achieve their competitive advantage in the competition in the effort-expending stage, the players in the group have an incentive to strongly motivate themselves to win the prize, and actually do so by choosing a "large" winner's fractional share. Such strategic behavior of the group leads to the finding that $\sigma_i^{ub} < \sigma_i^{ob}$. Next, the equilibrium total effort level is less in the case of unobservable sharing rules than in the case of observable sharing rules. Not surprisingly, this happens because, in the case of observable sharing rules, the players in a group choose a larger winner's fractional share – so that they are motivated to exert more effort – compared to the case of unobservable sharing rules. On the basis of this result, we may argue that, in rent-seeking contests, the unobservability of sharing rules reduces social costs, compared to the case of observable sharing rules. Another interesting comparative result is that $\pi_i^{ub} < \pi_i^{ob}$ holds in part (b), while $\pi_i^{ub} > \pi_i^{ob}$ holds in part (c). This can be explained as follows. The observability of the sharing rule, which enables the players in the group to make a strategic commitment for the competition in the subsequent effort-expending stage, is beneficial to them as compared with the case of unobservable sharing rules, when there is just one group. However, it is harmful to the players in the groups, when there is more than one group, due to their stiffer competition. Finally, parts (b) and (c) say that the equilibrium expected payoff for each singleton is greater in the case of unobservable sharing rules than in the case of observable sharing rules. This comes from the following facts. The first is that $\sigma_{N+1} = 1$ both in the case of unobservable sharing rules and in the case of observable sharing rules. The second is that, in the case of unobservable sharing rules, $\sigma_{N+1}$ is greater than the equilibrium winner's fractional share of any group, so that each singleton possesses a competitive advantage over the players in the groups; however, in the case of observable sharing rules, $\sigma_{N+1}$ may be less than the equilibrium winner's fractional share.
of some group. The third is that \( \sigma_i^{pb} < \sigma_i^{ob} \) for all \( i \), which implies that the singletons face less motivated opponents in the case of unobservable sharing rules than in the case of observable sharing rules.

5. The main model: Endogenous group formation

In the basic model we have considered so far, the number of groups, their sizes, and the number of singletons are exogenously given. In this section, we consider the main model in which the number of groups, their sizes, and the number of singletons are endogenously determined. The main model is the same as the basic model with the exception that, in the main model, the players decide first whether to form groups. In the main model, we assume that \( n \geq 2, N \geq 0, \) and \( 2 \leq m_i \leq n \). Recall from Section 2 that we may treat the singletons as the members of group \( N+1 \) whose winner's fractional share is unity and publicly known.

We formally consider the following game. At the start of the game, the players decide simultaneously and independently whether to form groups, knowing that the winner should share the prize with the other players in his group, if any. Next, after knowing the number of groups and their sizes, the players in each group jointly decide how to share the prize among themselves if one of them wins it. That is, they make a binding agreement on their sharing rule \( \sigma_i \). Next, all the players in the contest choose their effort levels simultaneously and independently. When the players choose their effort levels, the players in each group know the sharing rule of their own group; however, each group's sharing rule is unobservable to the players in the other groups and the singletons. Lastly, the winning player is determined, and the players in his group, if any, share the prize according to their sharing rule on which they agreed.

5.1. Equilibrium numbers of groups, group sizes, and numbers of singletons

To solve the game, we need to work backward. However, the players' decisions on their effort levels and on their groups' sharing rules, given the number of groups, the group sizes, and
the number of singletons, have already been considered in Section 3. Thus we are ready to consider the players' decisions on forming groups at the start of the game.

When deciding on forming groups, the players have perfect foresight about their subsequent decisions on the sharing rules of their groups to be formed and on their effort levels. This means that, at the start of the game, the players can compute their expected payoffs reported in Lemma 3.

The players decide simultaneously and independently whether to form groups. Thus, in equilibrium, no player has an incentive to individually move out of a group or to individually move in a group. Expressed differently, an equilibrium is immune to any unilateral deviation. We do not consider coordinated deviations by coalitions of players.

Let \( N^* \) represent the equilibrium number of groups, and let \( m^*_i \) the equilibrium size of group \( i \), for \( i = 1, \ldots, N^* \). Let \( m^*_{N^*+1} \) represent the equilibrium number of singletons. Using Lemma B2 in Appendix B, we obtain the left part of Table 2 and Proposition 2.

**Proposition 2.** (a) If the number of players in the contest is four or smaller, then group formation occurs. (b) If the number of players is five or greater, then group formation does not occur. (c) If the contest consists of two or three players, only the grand coalition occurs.

Proposition 2 is in line with the results in the literature on endogenous group formation in contests or conflicts. See, for example, Bloch (2012) for related discussions.

As shown in Table 2, if the number of players in the contest is four, then the equilibrium number of groups is not unique. In equilibrium, all four players in the contest join a single group; or there are two groups, each of which consists of two players; or the players compete individually to win the prize. It follows from Table 2 that, in equilibrium, every player belongs to one of the groups or all the players compete individually to win the prize.

When there is a small number of players in the contest, the grand coalition occurs. This is because each player in the grand coalition claims the share \( V/n \) of the prize—without
expending any effort – which is greater than his expected payoff resulting when he moves out of
the group. However, as the number $n$ of players increases, each player's share $V/n$ of the prize
in the grand coalition becomes smaller, whereas his expected payoff resulting when he moves
out of the group remains unchanged. Consequently, for $n \geq 5$, the grand coalition is not formed;
furthermore, no group is formed.

In the case of observable sharing rules, quite different results are obtained on the
equilibrium numbers of groups and the equilibrium sizes of the groups. Baik and Lee (2001)
show, for example, the following. The equilibrium number of groups is one for $n \leq 5$, but is not
unique for $n \geq 6$. Only the grand coalition occurs for $n = 2$ or 3, the equilibrium group sizes are
3 and 4 for $n = 4$ or 5, and the grand coalition does not occur for $n \geq 5$. When just one group is
formed in equilibrium, there may be players who do not belong to the group and, for $n \geq 6$, the
equilibrium group size equals the smallest of integers greater than half the number of players.
When more than one group is formed, every player belongs to one of the groups, and the
difference in equilibrium group size between any two groups is at most one.

Comparing the results of the two cases, those of unobservable and observable sharing
rules, we argue that the unobservability of sharing rules hinders the players from forming groups
because it nullifies strategic advantage that the players could gain if they formed groups and
announced (and committed to) their sharing rules. In the case of observable sharing rules, the
players in a group may discourage the outsiders by showing the outsiders that they have boosted
their incentives through their winner's fractional share greater than unity, whereas such strategic
behavior is not available to a singleton. This implies that forming a group is advantageous,
which leads to the conclusion that group formation is a rule in the case of observable sharing
rules. On the other hand, in the case of unobservable sharing rules, the players in a group cannot
affect the effort levels of the outsiders by the choice of their sharing rule. Given this inability,
they focus rather on reducing the negative externality among themselves by choosing their
winner's fractional share less than unity, which makes them less competitive in the effort-
expending stage. By contrast, the benefit to be a singleton is relatively large, especially if many
players participate in the contest. Accordingly, forming a group is not so attractive, and group formation is an exception in the case of unobservable sharing rules.

In the literature on the theory of endogenous coalition formation, the equilibrium coalition structures usually involve multiple coalitions, and the grand coalition is typically not an equilibrium coalition structure (see, for example, Bloch 1995; Yi 1998; Yi and Shin 2000). This is in contrast to Proposition 2. Examining the equilibrium structures of strategic alliances of firms, Morasch (2000) finds that all firms join a single alliance if the number of firms in the industry is four or smaller, but alliance structures involving two or more alliances are quite likely if the number of firms in the industry is five or greater. Clearly, the former is similar to Proposition 2, whereas the latter is in contrast to Proposition 2.

5.2. Equilibrium effort levels and expected payoffs

Let $Q^*$ represent the equilibrium total effort level in the game, and let $\pi^*$ represent each player's equilibrium expected payoff in the game. Using Lemmas 2 and 3, Table 1, the left part of Table 2, and Lemma B1, we obtain the right part of Table 2.

It is of interest that, in Table 2, the equilibrium total effort level, $Q^*$, increases while each player's equilibrium expected payoff, $\pi^*$, decreases, as the number of players, $n$, increases. Table 2 implies that, for any number of players in the contest, the equilibrium total effort level is less than the players' valuation $V$ for the prize—in other words, complete dissipation or overdissipation of the contested prize never occurs in this contest. Furthermore, Table 2 together with Lemma B1 implies that, for any number of players, the equilibrium total effort level does not exceed the total effort level resulting from the individual contest—in other words, endogenous group formation does not increase prize dissipation, as compared with the individual contest.

Table 2 together with Lemma B1 implies that, if $n = 2$ or 3, then each player's equilibrium expected payoff is greater than each player's expected payoff resulting from the individual contest; if $n = 4$, then it is greater than or equal to each player's expected payoff.
resulting from the individual contest; if \( n \geq 5 \), then it is equal to each player's expected payoff resulting from the individual contest. The explanations for this are straightforward. When there is a small number of players in the contest, the grand coalition occurs, so that the players share the prize equally without expending any effort. However, when \( n \geq 5 \), no group is formed, so that the players compete individually to win the prize. On the basis of the above result, we may argue that, given \( n \leq 4 \), endogenous group formation is beneficial to the players in the contest, as compared with the individual contest.

In the case of observable sharing rules, quite different results are obtained on the equilibrium total effort level and each player's equilibrium expected payoff. Baik and Lee (2001) show, for example, the following. First, when just one group is formed in equilibrium, the equilibrium total effort level is smaller than with the individual contest; when two groups are formed in equilibrium, it is equal to the total effort level resulting from the individual contest; however, when more than two groups are formed in equilibrium, it is greater than with the individual contest. Second, when just one group is formed in equilibrium, endogenous group formation is beneficial both to the group members and to the singletons, as compared with the individual contest; however, when more than two groups are formed in equilibrium, it is never profitable to any players in the contest.

6. Conclusions

We have studied contests in which \( n \) risk-neutral players, each as a member of a group or as a singleton, compete by expending irreversible effort to win a prize. The prize is awarded to one of the players. The probability that a player wins the prize depends on the effort levels of the \( n \) players. The winner shares the prize with the other players in his group, if any, and each group's sharing rule is unobservable to the players in the other groups and the singletons when the \( n \) players choose their effort levels. In the basic model, the number of groups, their sizes, and the number of singletons are exogenously given. We have formally considered the following game. First, the players in each group jointly make a binding agreement on their
sharing rule. The players in each group observe the sharing rule of their own group, but cannot observe the sharing rules of the other groups. The singletons cannot observe the sharing rule of any group. Next, all the players in the contest choose their effort levels simultaneously and independently. Finally, the winning player is determined, and the players in his group, if any, share the prize according to their sharing rule on which they agreed.

In Section 3, we have solved the game, and obtained the equilibrium sharing rules of the groups, the equilibrium effort levels of the players, the equilibrium total effort level, and the equilibrium expected payoffs for the players. We have shown the following. First, each group's equilibrium winner's fractional share is less than unity, which means that if a player in the group wins the prize, then the winner takes less than the prize. Second, each group's equilibrium sharing rule is independent of the players' valuation $V$ for the prize, and the equilibrium winner's fractional share of a larger group is less than that of a smaller one. Third, each group's equilibrium effort level and the equilibrium total effort level are independent of the number $n$ of players in the contest and the sizes of groups 1 through $N$, as long as changes in these do not change $N$ or $m_{N+1}$. Fourth, the equilibrium effort levels of the groups are the same regardless of the groups' different sizes, and thus each player in a larger group expends less effort than each player in a smaller one. Fifth, each player in a larger group has a less equilibrium expected payoff than each player in a smaller one. Finally, the equilibrium expected payoff for each singleton is greater than that for each player in any actual group.

In Section 4, we have compared the outcomes of the case of unobservable sharing rules with those of the case of observable sharing rules. We have shown that the two cases yield the same outcomes if there are one group and one singleton. We have shown that, if there is more than one singleton or more than one group, then each group's equilibrium winner's fractional share is less in the case of unobservable sharing rules than in the case of observable sharing rules; the equilibrium total effort level is less in the case of unobservable sharing rules than in the case of observable sharing rules; and the equilibrium expected payoff for each singleton is greater in the case of unobservable sharing rules than in the case of observable sharing rules.
In Section 5, we have considered the main model in which the number of groups, their sizes, and the number of singletons are endogenously determined. We have shown the following. If the contest consists of two or three players, then the grand coalition occurs in equilibrium. If $n = 4$, then the grand coalition occurs; or there are two groups, each of which consists of two players; or the players compete individually to win the prize. If $n \geq 5$, then the individual contest occurs—that is, group formation does not occur. We have also shown the following. For any number of players in the contest, the equilibrium total effort level is less than the players' valuation $V$ for the prize and, further, does not exceed the total effort level resulting from the individual contest. If $n \leq 4$, then endogenous group formation is beneficial to the players in the contest, as compared with the individual contest. Overall, we have obtained quite different results from those in Baik and Lee (2001), which considers the case of observable sharing rules.

In Section 5, to obtain the equilibrium numbers of groups, the equilibrium sizes of the groups, and the equilibrium numbers of singletons, we have examined only whether a player has an incentive to individually move out of a group or to individually move in a group. What happens if we use the concept of coalition-proof Nash equilibrium which takes into account also coordinated deviations by coalitions of players? In this case, we obtain the following sharper predictions. If $n \leq 4$, then only the grand coalition occurs in equilibrium. If $n \geq 5$, then no group is formed and the individual contest occurs in equilibrium.

In the models that we have considered in this paper, private information on sharing rules is exogenously assumed. Specifically, we have assumed that each group's sharing rule is unobservable to the players in the other groups and the singletons, when the $n$ players choose their effort levels. However, we may well expect that each group has the option of making its sharing rule observable or unobservable. Thus it would be interesting to consider a model that incorporates groups' decisions on whether they will make their sharing rules observable or unobservable. Next, in the models that we have considered in this paper, the prize is awarded to one of the players in the contest, and even the players in the same group compete with each other
to win the prize. It would be interesting to consider a model in which the players in each group pool their effort to win the prize, the prize is awarded to one of the singletons or one of the groups, and the players in the winning group, if any, share the prize among themselves.  

Finally, we have assumed in Section 5 that, at the start of the game, the players decide simultaneously and independently whether to form groups. It would be interesting to consider a model in which the players decide sequentially whether to form groups and, if a player decides to join an existing group, the existing group may block the player from joining it. We leave these modifications and extensions for future research.
Footnotes


2. We can think of some ways of the groups being committed to their sharing rules. For example, the contest organizer or the decision-maker who has authority to select the winner can make the groups committed to their sharing rules by enforcing the groups to adhere to their announced sharing rules.

3. See Section 6 for a discussion of a model in which the prize is awarded to one of the singletons or one of the groups.

4. Specific examples for such an agreement come from college football and from competition among university professors and researchers for research grants, which have been introduced in Section 1. Another example comes from competition among researchers for federal research grants. Many states in the United States have a matching grant program for researchers who secure federal grants.

5. In this respect, this paper is related to the literature on contests with externalities (see, for example, Esteban and Ray 1999; Kolmar and Wagener 2013).

6. See Baik and Lee (2007) for a general model of the simultaneous-move game with sequential moves.

7. The decision of the players in group $i$ on their sharing rule is not directly affected by their beliefs about the other groups' sharing rules. This is because their expected payoffs do not depend directly on the other groups' sharing rules and because they play a simultaneous-move
game with the players in the other groups and the singletons. Note, however, that their expected payoffs depend \textit{indirectly} on the other groups' sharing rules.

8. The second-order condition is satisfied for every maximization problem in this paper. For concise exposition, however, we do not state it explicitly in each case.

9. Because the players in group $i$ will choose the same effort level, $x_i(\sigma_i, Q_{-i})$, they have the same expected payoff, $\pi_i(\sigma_i, Q_{-i})$.

10. Note that $\pi_i(\sigma_i, Q_{-i}) > 0$ if and only if $V > Q$.

11. Baik and Lee (2001) obtain the same result in the case of observable sharing rules. Nitzan and Ueda (2011) study collective contests in which each group's sharing rule is endogenously determined and is unobservable to the players in the other groups, and identify the cases where a larger group chooses a more egalitarian sharing rule than a smaller group.

12. In the literature on the theory of contests, a standard result is that the equilibrium total effort level increases as the number of players increases.

13. An observation similar to this is obtained in contests with group-specific public-good prizes (see, for example, Baik 2008).

14. Recall the observation, stated in the preceding paragraph, that $Q_i(m)$, for $i = 1, \ldots, N$, and $x_{N+1}(m)$ are equal to the corresponding players' equilibrium effort levels obtained in the reduced contest with only $(N + m_{N+1})$ players.

15. This implies that $Q_h(m)/Q(m) = Q_t(m)/Q(m)$ holds for any $h$ and $t$ with $h, t = 1, \ldots, N$. That is, in equilibrium, the probability that any one of the players in group $h$ wins the prize is the same as the probability that any one of the players in group $t$ wins it, regardless of the groups' sizes. Considering a noncooperative rent seeking contest with quadratic costs in which players form groups sequentially, Bloch et al. (2006) show that the equilibrium winning probability is the same across groups. By contrast, Esteban and Ray (2001) show, in a model with nonlinear lobbying costs, that the equilibrium winning probability of a larger group is higher than that of a smaller one. Nitzan and Ueda (2011) show that the equilibrium winning probability of a larger group is higher than that of a smaller one, unless the prize is purely private.
16. Esteban and Ray (2001) show, in a model with nonlinear lobbying costs, that the equilibrium expected payoff to a player increases with group size when the collective good is purely public, and decreases with group size when the collective good is purely private.

17. We can readily understand this by recalling the observation, stated above, that $Q_i(m)$, for $i = 1, \ldots, N$, and $x_{N+1}(m)$ are equal to the corresponding players' equilibrium effort levels obtained in the reduced contest with only $(N + m_{N+1})$ players.

18. Note that, if values of $V$, $m_i$, $N$, and $m_{N+1}$ are given, then $Q_i^{ub}$ and $\pi_i^{ub}$ remain unchanged while $Q_i^{ob}$ and $\pi_i^{ob}$ change, as $n$ increases. Thus $Q_i^{ub} > Q_i^{ob}$ may hold for some $i$, but not all $i$, if $n$ is large; $\pi_i^{ub} < \pi_i^{ob}$ may hold for some $i$, but not all $i$, if $n$ is sufficiently large. We find that $Q_i^{ub} > Q_i^{ob}$ may hold for some $i$, if $n \geq 7$, and that $\pi_i^{ub} < \pi_i^{ob}$ may hold for some $i$, if $n \geq 10$.

19. The following are additional information regarding group $i$'s equilibrium winner's fractional share. Recall from Lemma 1 that $1/m_i < \sigma_i^{ub} < 1$ holds always. By contrast, Baik and Lee (2001) show in the case of observable sharing rules that $1/m_i < \sigma_i^{ob} < 1$ holds if $m_i > n/2$; $\sigma_i^{ob} = 1$ holds if $m_i = n/2$; and $\sigma_i^{ob} > 1$ holds if $m_i < n/2$.

20. Baik and Lee (2001) show in the case of observable sharing rules that $\sigma_i^{ob} > 1$ holds if $m_i < n/2$.

21. Different procedures can be designed for the players' simultaneous decisions on forming groups at the start of the game. One possible procedure is that the players first talk one another about forming groups, and then they decide simultaneously and independently whether to join the potential groups. The following is another possible, but more formal, procedure. Each player throws his name tag into one of $n$ jars. If more than one player throws their name tags into the same jar, they form a group. If only one player throws his name tag in one of the jars, he does not belong to any group; instead, he becomes a singleton. In order to facilitate group formation, one player may be allowed to throw his name tag in one of the jars before the other players do.
22. See Section 6 for a discussion of using the concept of coalition-proof Nash equilibrium, introduced by Bernheim et al. (1987), to obtain the equilibrium numbers of groups, the equilibrium sizes of the groups, and the equilibrium numbers of singletons.

23. Bloch et al. (2006) show that the grand coalition is a unique equilibrium coalition structure in the noncooperative rent seeking contest with quadratic costs in which players form groups sequentially.

24. By the case of observable sharing rules, we mean the game which is the same as the one in Section 5 with the exception that each group's sharing rule is now observable to all the players in the game.

25. However, if coalition formation among symmetric players creates negative externalities for singletons, the grand coalition is an equilibrium coalition structure under the open membership rule.

26. In this modified model, one may assume that the probability that group $i$ wins the prize is $X_i/X$ for $X > 0$, and it is $1/(N + m_{N+1})$ for $X = 0$. One may use the following sharing-rule specification which is used in Nitzan (1991a, 1991b), Baik and Shogren (1995), Hausken (1995), Lee (1995), Davis and Reilly (1999), Ueda (2002), Baik and Lee (2007), Nitzan and Ueda (2011), and Ursprung (2012): The fractional share $\lambda_{jk}$ of player $k$ in the winning group, say group $j$, is

$$\lambda_{jk} = \theta_j x_{jk} / X_j + (1 - \theta_j) / m_j,$$

where the parameter $\theta_j$ is chosen by the players in group $j$ before they choose their effort levels. Note that the players in group $j$ need to know how much effort each player expended when they share the prize. Based on the results in Baik et al. (2006), one may conjecture that the modified model with this sharing-rule specification yields the same (main) results as in the current paper.
Appendix A: Obtaining the players' equilibrium effort levels

From equations (8) and (9), we have the following system of $N+1$ simultaneous equations:

\begin{align*}
Q^2 &= V(m_2x_2 + m_3x_3 + \cdots + m_Nx_N + m_{N+1}x_{N+1}) \\
Q^2 &= V(m_1x_1 + m_3x_3 + \cdots + m_Nx_N + m_{N+1}x_{N+1}) \\
&\vdots \\
Q^2 &= V(m_1x_1 + m_2x_2 + \cdots + m_{N-1}x_{N-1} + m_{N+1}x_{N+1})
\end{align*}
and

\begin{align*}
Q^2 &= V\{m_1x_1 + m_2x_2 + \cdots + m_Nx_N + (m_{N+1} - 1)x_{N+1}\}.
\end{align*}

These equations can be rewritten as

\begin{align*}
Q^2 &= V(Q - m_1x_1) \\
Q^2 &= V(Q - m_2x_2) \\
&\vdots \\
Q^2 &= V(Q - m_Nx_N)
\end{align*}
and

\begin{align*}
m_{N+1}Q^2 &= V(m_{N+1}(Q - m_{N+1}x_{N+1}).
\end{align*}

Adding these equations together, we have

\begin{align*}
(N + m_{N+1})Q^2 &= V\{(N + m_{N+1})Q - Q\}.
\end{align*}

This yields

\begin{align*}
Q &= V(N + m_{N+1} - 1)/(N + m_{N+1}).
\end{align*}

Substituting this expression for $Q$ into the equations in (A1), we obtain the players' equilibrium effort levels, $x_1(m)$ through $x_{N+1}(m)$. 
Appendix B: Proof of Proposition 2

Lemma B1, which can be easily proved, is useful in obtaining Lemma B2 below and the right part of Table 2.

Lemma B1. (a) If a grand coalition is formed (i.e., \( m_1 = n \)), then the equilibrium sharing rule of the group is \( 1/n \), each player in the group chooses zero effort levels, and each player's equilibrium expected payoff is \( V/n \). (b) If the players compete individually to win the prize (i.e., \( N = 0 \)), then each player chooses an effort level of \( V(n - 1)/n^2 \), the equilibrium total effort level is \( V(n - 1)/n \), and each player's equilibrium expected payoff is \( V/n^2 \).

Using Lemma 3 and Lemma B1, we obtain Lemma B2, which is useful in obtaining Proposition 2.

Lemma B2. (a) If the difference in group size between two actual groups is two or greater, then a player in the larger group has an incentive (to move out of the larger group and) to move in the smaller group. Otherwise, no player in one group has an incentive to move in the other group. (b) Suppose that \( N \geq 2 \). If \( N = 2, m_1 = m_2 = 2, \) and \( m_{N+1} = 0 \), then no player in the groups has an incentive to be a singleton. Otherwise, there is at least one player in the groups who has an incentive to be a singleton. (c) Suppose that \( N = 1 \) and \( m_{N+1} \geq 1 \). If \( m_1 = 2 \) and \( m_{N+1} = 1 \), then no player in the group has an incentive to be a singleton. Otherwise, a player in the group has an incentive to be a singleton. (d) Suppose that a grand coalition is formed. If \( m_1 = n = 2, 3, \) or \( 4 \), then no player in the group has an incentive to be a singleton. Otherwise, a player in the group has an incentive to be a singleton. (e) Suppose that \( N \geq 1 \) and \( m_{N+1} \geq 1 \). If \( N = 1, m_N = 2, \) and \( m_{N+1} = 1 \), then the singleton has an incentive to move in group \( N \). Otherwise, no singleton has an incentive to move in group \( N \). (f) Suppose that \( N = 0 \). If \( m_{N+1} = n = 2 \) or \( 3 \), then a singleton has an incentive to unilaterally deviate (by throwing his
name tag into another player's jar). Otherwise, no singleton has an incentive to unilaterally deviate.

The proof of Lemma B2 is straightforward and therefore omitted. In proving part (a), for example, we compare the expected payoff for each player in one group – computed before a player moves out of the group – with the expected payoff for each player in the other group resulting when a player moves out of the first group and moves in the second group. Note that, in Lemma B2, a player has an incentive to move out of a group and/or to move in a group if such a movement increases his expected payoff.
References


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<th>Unobservable Sharing Rules</th>
<th>Observable Sharing Rules</th>
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<td>Group $i$'s Sharing Rule</td>
<td>$1 - \frac{m_i - 1}{m_i(N + m_{N+1})}$</td>
<td>$1 + \frac{n - 2m_i}{m_i {n(N-1) + 2m_{N+1}}}$</td>
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<tr>
<td>Effort Level of Each Player in Group $i$</td>
<td>$\frac{V(N + m_{N+1} - 1)}{m_i(N + m_{N+1})^2}$</td>
<td>$\frac{V(n - m_i) {n(N-1) + 2m_{N+1} - 1}}{m_i {n(N-1) + 2m_{N+1}}^2}$</td>
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<td>Effort Level of Each Singleton</td>
<td>$\frac{V(N + m_{N+1} - 1)}{(N + m_{N+1})^2}$</td>
<td>$\frac{V {n(N-1) + 2m_{N+1} - 1}}{{n(N-1) + 2m_{N+1}}^2}$</td>
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<tr>
<td>Group $i$'s Effort Level</td>
<td>$\frac{V(N + m_{N+1} - 1)}{(N + m_{N+1})^2}$</td>
<td>$\frac{V(n - m_i) {n(N-1) + 2m_{N+1} - 1}}{{n(N-1) + 2m_{N+1}}^2}$</td>
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<td>Singletons' Total Effort Level</td>
<td>$\frac{V m_{N+1} {N + m_{N+1} - 1}}{(N + m_{N+1})^2}$</td>
<td>$\frac{V m_{N+1} {n(N-1) + 2m_{N+1} - 1}}{{n(N-1) + 2m_{N+1}}^2}$</td>
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<td>Total Effort Level</td>
<td>$V(1 - \frac{1}{N + m_{N+1}})$</td>
<td>$V \left[1 - \frac{1}{n(N-1) + 2m_{N+1}}\right]$</td>
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<td>Expected Payoff for Each Player in Group $i$</td>
<td>$\frac{V}{m_i(N + m_{N+1})^2}$</td>
<td>$\frac{V(n - m_i)}{m_i {n(N-1) + 2m_{N+1}}^2}$</td>
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<td>Expected Payoff for Each Singleton</td>
<td>$\frac{V}{(N + m_{N+1})^2}$</td>
<td>$\frac{V}{{n(N-1) + 2m_{N+1}}^2}$</td>
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TABLE 2

The Equilibrium Numbers of Groups, Group Sizes, Numbers of Singletons, Effort Levels, and Expected Payoffs

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<tr>
<th>$n$</th>
<th>$N^*$</th>
<th>$m_1^*$</th>
<th>$m_2^*$</th>
<th>$m_{N^<em>+1}^</em>$</th>
<th>$Q^*$</th>
<th>$\pi^*$</th>
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<tr>
<td>2</td>
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<td>2</td>
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<td>0</td>
<td>0</td>
<td>$V/2$</td>
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<td>3</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<tr>
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<td>1</td>
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<td>0</td>
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<td>2</td>
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<td>2</td>
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<td>$V/2$</td>
<td>$V/8$</td>
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<tr>
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<td>$V/16$</td>
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<tr>
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<td></td>
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<tr>
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<td>$s$</td>
<td></td>
<td>$V(s - 1)/s$</td>
<td>$V/s^2$</td>
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<td>$s$</td>
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