Incomplete Markets as the Outcome of Bilateral Bargaining*

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Abstract

To complement the theory of incomplete markets under perfect competition and anonymity, this paper examines the theory of incomplete markets under strategic bargaining. Households bargain over bilateral nominal contracts that specify transfers for all states of uncertainty. The lone institutional feature is the limit on the number of contracts that a household can agree to. These contract limits for all households determine whether or not the equilibrium allocations are first-best. Specifically, if all households can be linked via a series of contracts to all other households, then the equilibrium allocations will be first-best.

Keywords incomplete markets – bargaining – strategic foundations – constrained suboptimality – regularity – asymmetric information

JEL Classification D52 · D82 · E21 · G11

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1 Introduction

One of the significant achievements by economists in the past 30 years is a broad understanding of the outcomes of decision-making in competitive markets with an uncertain future. With research in the field of finance trending toward a greater use of heterogeneous agent models and macroeconomists also making the switch to general equilibrium models, variants of the GEI financial model are becoming widely used. The equilibrium properties when financial contracts are competitively traded are well-known (see Magill and Quinzii [1996]), but very little is known if competitive trade is replaced with strategic bargaining. This paper demonstrates that, when households bargain over bilateral contracts, the normative results are in line with those attained for competitive markets. In particular, if the institutional primitives allow all households to be linked via a series of contracts, then the equilibrium allocations will be Pareto optimal.

Researchers often view competitive markets as idealized mechanisms and have sought to justify their use by relaxing the price-taking assumption. This assumption is a feature of the institutions buttressing the model. At a more fundamental level lie the opportunities and choices of households. In the static pure-exchange model, conditions have been found so that the outcome when households bargain over commodity contracts is equivalent to the competitive outcome (see Feldman [1973], Goldman and Starr [1982], Gale [2000], Yildiz [2003], Dávila and Eeckhout [2008], and Penta [2011]). This paper considers a dynamic setting where households bargain over financial contracts. In place of institutions specifying asset payouts and asset prices, the model only requires a limit on the number of contracts that a household can agree to.

Other than the strategic foundations of competitive equilibria, a handful of broad attempts have been made to blend a competitive environment with strategic behavior of households. Examples include club theory (see Ellickson et al. [1999] and Ellickson et al. [2001]) and strategic market games (see Shapley and Shubik [1969] and Giraud [2003]). In all cases, economists incorporate the broad notion that a household's trading opportunities can be impacted by available actions or information lying outside the competitive paradigm. These fields and others (on the topic of incomplete markets, see Yildiz [2002] and Oksendal and Sulem [2008]) share with the current paper the question of whether the outcomes of a more realistic strategic setting are commensurate with the competitive outcomes.

Allowing the model dynamics to occur over two periods with a finite number of states of uncertainty in the second, this paper removes the exogenous asset structure as the means of wealth transfer for households. Rather, households bargain over bilateral nominal contracts specifying transfers for all states of uncertainty. With nominal contracts and a fixed asset
structure, it is known that GEI equilibria are generically indeterminate (see Cass [1992]). Yet, when households exchange nominal contracts and are not constrained by a fixed asset structure (as is the case with this model), I show that equilibria are generically determinate.

In a sense, each household is selecting, subject to its contract limit, individual assets to trade. This brings into the discussion the literature on financial innovation, where an outside agent (typically called the innovator) sets the asset structure in order to maximize asset trade. The literature is split into two approaches. The first approach is Allen and Gale (1988) who show that the equilibrium allocations that result after financial innovation are constrained optimal, but not Pareto optimal.\(^1\)

The second approach is Pesendorfer (1995) whose model presumes a set of standard securities with trading costs. Innovators then create new securities from the standard ones and seek to market them (at a cost) to investors in competitive markets. With only one innovator, as the trading and marketing costs approach zero, the equilibria become determinate and their respective allocations become Pareto optimal. In my model, individual contracts are traded, yet equilibria remain determinate and the contract limits determine in a clean way whether the equilibrium allocations are Pareto optimal or generically constrained Pareto suboptimal.

Though not a model of financial innovation, the Kiyotaki and Wright model (1991) shows that households may choose to trade intrinsically worthless fiat money, because it can reduce the cost of search frictions. Thus, search frictions are the reason why trade may occur via a monetary transaction, where a monetary transaction is nothing other than a bilateral nominal contract. In my paper, I utilize nearly identical bilateral nominal contracts (albeit in a different bargaining environment) not as an explanation for money, but rather as an explanation for incomplete risk-sharing opportunities. Thus, instead of search frictions, the institutional frictions in my model are the contract limits.

In line with the contract limits, restrictions need to be imposed on the bargaining process. Otherwise, households could negotiate contracts that satisfy the contract limits, yet provide transfers achievable under complete risk-sharing. The restrictions are twofold. First, during the bargaining process, each pair of households meets only one time. Second, bargaining within a pair proceeds using only "take it or leave it" offers.

For the majority of the paper, I analyze a complete information setting. Included in the information common to all households is the knowledge about which household in a pair has been chosen to make the "take it or leave it" offer. The bargaining occurs in two stages: first, all households simultaneously make contract proposals; second, each household decides

\(^1\)The follow-up paper Allen and Gale (1991) shows that with imperfect competition, as is required to prove existence without imposing short sale constraints, allocations are not even constrained optimal.
which contract proposals to accept subject to its contract limit. With complete information, it is equivalent to view this first stage as publicly announced contract proposals. This entire bargaining process takes place in the initial period.

A branch is defined as a set of households that are linked via a series of contracts. This paper proves that if it is possible for all households to belong to one branch (keep in mind that the contract limits may or may not permit this), then it is optimal for the households to offer contracts so that one branch is formed. This leads to the first normative result of the paper: if all households can be linked via a series of contracts to all other households, then the resulting equilibrium allocations will be Pareto optimal. A constrained Pareto optimal allocation is an allocation that cannot be Pareto improved using only transfers between households in the same branch. The second normative result of the paper states that if the households cannot all be linked via a series of contracts and multiple commodities are traded, then the equilibrium allocations will be generically constrained Pareto suboptimal.

This paper is organized into six remaining sections. In Section 2, I pair the contracting environment with a standard two-period general equilibrium model. In Section 3, I consider the conditions for the Pareto optimality property. In Section 4, I consider the conditions for the properties of Pareto suboptimality and constrained Pareto suboptimality. In Section 5, I examine the impact of asymmetric information on the equilibrium outcome. Section 6 concludes and Section 7 contains the proofs of the main results.

2 The Model

Consider a two-period general equilibrium model with $S$ states of uncertainty in the second period. Denoting the first period as the $s = 0$ state, I number the states as $s \in S = \{0, ..., S\}$. In each state, a finite number of households $h \in H = \{1, ..., H\}$ trade and consume a finite number ($L < \infty$) of physical commodities. The commodities are denoted by the variable $x$. Define the total number of goods as $G = L(S + 1)$. Concerning notation, the vector $x^h \in \mathbb{R}^G_+$ contains the entire consumption by household $h$, the vector $x^h(s) \in \mathbb{R}^L_+$ contains the consumption by household $h$ in state $s$ (of all commodities), and the scalar $x^h_{l}(s) \in \mathbb{R}_+$ is the consumption by household $h$ of the good $(s, l)$, or the $l^{th}$ physical commodity in state $s$.

Households are endowed with commodities in all states. These endowments are denoted by $e = (e^h)_{h \in H} \in \mathbb{R}^{HG}_+$. I assume that all households have strictly positive endowments:

**Assumption 1** $e^h >> 0 \ \forall h \in H$.\footnote{The vector notation $y >> 0$ means that $y_i > 0 \ \forall i$, whereas $y \geq 0$ means $y_i \geq 0 \ \forall i$, and $y > 0$ means $y_i > 0 \ \forall i$.}
Define the set of endowments satisfying Assumption 1 as $\mathcal{E} = \{(e^h)_{h \in H} : e^h >> 0\}$. The consumption set $X^h$ is assumed to be the interior of $\mathbb{R}^G_+$:

**Assumption 2**

$X^h = \mathbb{R}^G_{++} \forall h \in H$.

The utility function $u^h : X^h \rightarrow \mathbb{R}$ is subject to the following assumptions:

**Assumption 3**

$u^h$ is $C^3$, differentiably strictly increasing (i.e., $Du^h(x^h) >> 0 \forall x^h \in X^h$), differentiably strictly concave (i.e., $D^2u^h(x^h)$ is negative definite $\forall x^h \in X^h$), and satisfies the boundary condition $(clU^h(x^h) \subset X^h$ where $U^h(x^h) = \{x' \in X^h : u^h(x') \geq u^h(x^h)\}$) $\forall h \in H$.

Define the set of utility functions satisfying Assumption 3 as $\mathcal{U} = \{(u^h)_{h \in H} : u^h$ satisfies Assumption 3$\}$.

The commodity markets are Walrasian, so I introduce the commodity prices $p \in \mathbb{R}^G \backslash \{0\}$.

Under Assumption 3, the prices satisfy $p \in \mathbb{R}^G_{++}$.

### 2.1 The contracting environment

The commodities are perishable, so households require a means by which to transfer wealth between states. In this paper, that means is provided through bilateral nominal contracts. The terms of the contracts are reached using "take it or leave it" bargaining offers.

The entire bargaining process (contract offers and acceptance decisions) occurs in the initial period. Bilateral bargaining requires two households, but in this model (unlike Gale [2000]), a pair has only a single opportunity for bargaining. In that single opportunity, one household makes a "take it or leave it" offer and the other household decides to either accept or reject that offer.

In the complete information setting, this single-meeting environment is best interpreted as public contract offers. The key feature governing the contract offers and acceptance decisions is the "contract limit" of a household. For each household $h \in H$, $\eta^h$ is the "contract limit," that is, the total number of contracts that $h$ can enter into. I define an economy as the parameters $(e^h, u^h, \eta^h)_{h \in H}$. The following assumption is made for convenience:

**Assumption 4** $\eta^h \geq 1 \ \forall h \in H \text{ and } \sum_{h \in H} \eta^h$ is even.

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$y \geq 0$ and $y \neq 0$.

3These Walrasian markets can equivalently be viewed as the outcome of a bilateral bargaining process under certain conditions (see Gale [2000]).

4The time horizon for the dynamic bargaining process of the model is far smaller than the time horizon for the random selection of a state of uncertainty.
Consider a contracting pair \((h, h')\). Recall that any pair has only the single opportunity for bargaining, so one household makes the "take it or leave it" offer and the other household either accepts or rejects the offer. The following rule governs which household in a pair makes the "take it or leave it" offer:

**Assumption 5** For all \(h \in \mathcal{H}\), the household \(h\) is the one in the pair \((h, h')\) to make the "take it over leave it" offer, whenever \(h' > h\).

This feature of the model specifies that household \(h = 1\) makes "take it or leave it" offers in all pairs \((1, h')\) for \(h' > 1\), household \(h = 2\) makes "take it or leave it" offers in all pairs \((2, h'')\) for \(h'' > 2\), and so forth. This household labeling is without loss of generality, though certainly generality is lost with the transitivity requirement in Assumption 5. The labeling of households is common knowledge.

**Remarks on Assumption 5**

To impose structure on the model and avoid "anything goes" results, a clear ordering of bargaining power is required. To justify Assumption 5, consider that the institutions on the periphery of the economic model that permit household \(h\) to make a contract offer to \(h'\) and household \(h'\) to make a contract offer to \(h''\) are also likely to permit household \(h\) to make a contract offer to \(h''\). This transitivity requirement holds only if Assumption 5 does. Clearly, restricting bargaining to only "take it or leave it" offers is a limitation of the model, but this choice is made to simplify the equilibrium characterization process. See Section 6 for comments about generalizing the bargaining process.

The contract offers are made simultaneously. After all contract offers are made, households decide which of the offers to accept. It is irrelevant whether the contract acceptance choices occur simultaneously or sequentially.

The notational convention is that for any pair \((h, h')\) with \(h' > h\), the contract proposal by \(h\) is \(\tilde{\gamma}(h, h') \in \mathbb{R}^{S+1}\). The proposed contract specifies that household \(h\) receives the nominal transfer of \(+\tilde{\gamma}_s(h, h')\) in state \(s \in S\), provided the proposal is accepted. The agreed-upon contract between the proposer \(h\) and the receiver \(h'\) is \(\gamma(h, h')\). If \(h'\) accepts the proposal \(\tilde{\gamma}(h, h')\), then \(\gamma(h, h') = \tilde{\gamma}(h, h')\). If \(h'\) rejects the proposal, then \(\gamma(h, h') = 0\). For consistency (zero net transfers), I require that the accepted contract \(\gamma(h, h')\) specifies that \(h'\) will receive the nominal transfer of \(-\gamma_s(h, h')\) in state \(s \in S\).

Recall that household \(h\) proposes contracts in all pairs \((h, h')\) with \(h' > h\). Define \(\tilde{\gamma}^h = (\tilde{\gamma}(h, h'))_{h' > h}\) as the contract proposals by household \(h\) and \(\gamma^h = (\gamma(h, h') : \gamma(h, h') = \tilde{\gamma}(h, h'))_{h' \neq h}\) as the accepted contracts from household \(h\). The contract \(\tilde{\gamma}(h, h') = 0\) is the trivial contract. When determining whether or not a household violates its contract limit, I am only
concerned with nontrivial contracts. The "contract limit" constraint \((CL)\) for household \(h\) is:

\[
\sum_{h' > h} 1 \{ \gamma(h, h') \neq 0 \} + \sum_{h' < h} 1 \{ \gamma(h', h) \neq 0 \} \leq \eta^h.\tag{CL}
\]

In this environment, all contracts are fully committed to. In all states (including the initial period), after the nominal transfers specified by the contracts are made, the spot markets operate as Walrasian markets. To be clear, the timing of actions in the initial period proceeds in the following order:

0. The ordering of households is randomly determined.
1. Contracts are simultaneously proposed: for any household \(h\), taking as given the commodity prices \(p\), the contract proposals \((\tilde{\gamma}^h)'_{h' \neq h}\), the household parameters \((e^h, u^h, \eta^h)_{h \in \mathcal{H}}\), and the ordering of households, household \(h\) proposes \(\tilde{\gamma}^h\).
2. Observing all contract proposals, households either accept or reject the contract proposals made to them.
3. Households make initial period consumption choices \((x^h(0))_{h \in \mathcal{H}}\).

### 2.2 An unconstrained financial equilibrium

To define the initial equilibrium concept, I first define the household problem. For simplicity, any contract \(\tilde{\gamma}(h, h') \in \mathbb{R}^{S+1}\) is a column vector and \(P = \begin{bmatrix} p(0) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & p(S) \end{bmatrix}\) is the \((S+1) \times G\) price matrix. For household \(h\), taking as given the variables \(p\) and \((\tilde{\gamma}^h)'_{h' \neq h}\), the household problem \((HP)\) is given as follows:

\[
\max_{x^h \in \mathcal{X}^h, \tilde{\gamma}^h \in \mathcal{R}^{(H-h) \times (S+1)}} u^h(x^h)
\quad \text{subj. to}
\begin{align*}
1. & \quad P(e^h - x^h) + \sum_{h' > h} \gamma(h, h') - \sum_{h' < h} \gamma(h', h) \geq 0. \tag{HP}
2. & \quad \text{Constraint (CL) holds.}\tag{6}
\end{align*}
\]

Given the household problem \((HP)\), I define an **unconstrained financial equilibrium**.

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5 The indicator function \(1\{p\}\) takes the value 1 when the statement \(p\) is true and 0 otherwise.

6 The budget constraint and the constraint \((CL)\) are not only functions of household \(h\) decisions; they both also depend on the accept/reject decisions of the other households. The choice of household \(h\) is invalid if either constraint is violated, where an invalid choice results in an infinitely large utility penalty.
Definition 1 \((x^h, \tilde{\gamma}^h, \gamma^h)_{h \in H, p}\) is an unconstrained financial equilibrium if

1. \(\forall h \in H\), given \((p, (\tilde{\gamma}^{h'})_{h' \neq h})\),
   \((x^h, \gamma^h)\) is an optimal solution to the household problem \((HP)\),

2. for any eligible pair \((h, h')\) with \(h < h'\), \(\gamma (h, h') = \tilde{\gamma} (h, h')\) if \(h'\) accepts the proposal,

3. markets clear:
   \[
   \sum_{h \in H} (x^h_l(s) - e^h_l(s)) = 0 \quad \forall (l, s).
   \]

This definition of an unconstrained financial equilibrium is equivalent to (i) a subgame perfect Nash equilibrium over the two-stage game with contract offers in the first stage and acceptance decisions in the second stage and (ii) a Walrasian equilibrium over each of the spot markets \(s \in S\).

2.3 Acceptance condition

In order for the contract proposal \(\tilde{\gamma} (h, h')\) (a choice variable of \(h\)) to be included in the budget constraint of \(h\), the household must take into account the Acceptance condition of household \(h'\). The following paragraphs introduce the notation required to specify the Acceptance condition, which states that any household \(h'\) will accept the contract proposal \(\tilde{\gamma} (h, h')\) from any \(h < h'\).

Define a household’s total nominal payoff from contracts in state \(s \in S\) as \(\tau^h(s) = \sum\limits_{h' > h} \gamma^h (h, h') - \sum\limits_{h' < h} \gamma^h (h', h)\) with \(\tau^h = (\tau^h(s))_{s \in S}\). For \(h\), define \(\bar{\Gamma}_h : \mathbb{R}^{(H-h)(S+1)} \rightarrow \mathbb{R}^{(H-h)(S+1)}\) as the best response (in terms of contract proposals) for household \(h\), taking as given the proposed contracts of all other households \((\tilde{\gamma}^{h'})_{h' \neq h}\).

Define \(\sigma^h : \mathbb{R}^{(S+1)} \rightarrow X^h\) as the consumption demand function given the transfers \(\tau^h \in \mathbb{R}^{(S+1)} : x^h = \sigma^h (\tau^h)\). The function also depends on the commodity prices \(p\) and on the parameters of the model, notably endowments \(e^h\), but these are omitted for simplicity.

For the pair \((h, h')\) with \(h < h'\), define \(\tilde{\gamma} (\backslash h, h') = (\tilde{\gamma} (h'', h'))_{h'' < h', h'' \neq h}\) as the proposals received by \(h'\) from households other than \(h\) and \(\tilde{\gamma} (\backslash h, \backslash h') = (\tilde{\gamma}^{h''} (h'' \notin \{h, h'\}), (\tilde{\gamma} (h, h''))_{h'' \neq h'})\) as all contract proposals by households different than \(h'\), except \(\tilde{\gamma} (h, h')\). Define the \((S + 1) \times (H - h)\cdot (S + 1)\) matrix \(M^h = [I_{S+1} ... I_{S+1}]\) and the \((S + 1) \times (h - 1)\cdot (S + 1)\) matrix \(N^h = [I_{S+1} ... I_{S+1}]\). The Acceptance condition \((AC)\) for a contract proposal from \(h\) to \(h'\) can then be stated as:
\[
\tilde{\gamma}(h, h') = \gamma(h, h') \text{ iff } \\
u^h \left\{ \sigma^{h'} \left[ M^{h'} \cdot \tilde{\Gamma}^{h'} (\tilde{\gamma}(\mathcal{h}, \mathcal{h'}), \tilde{\gamma}(h, h')) - N^{h'} \cdot \left( \tilde{\gamma}(\mathcal{h}, h') \right) \right] \right\} \geq (AC)
\]

The best response \( \tilde{\Gamma}^{h'} (\tilde{\gamma}(\mathcal{h}, \mathcal{h'}), \tilde{\gamma}(h, h')) \) is the maximizer (in terms of contract proposals) of the household problem \((HP)\) for household \(h'\). As written, \(h'\) expects that it will accept all contract proposals and that each of its contract proposals will be accepted. This is because all households make their contract proposals subject to \((AC)\) above. First, I will define an equilibrium with this restriction in the household problem \((HP)\) of all households. Then I will show that the decision by households to make contract proposals subject to \((AC)\) is not a restriction, but an equilibrium choice.

### 2.4 A financial equilibrium

The main equilibrium concept of this paper is a financial equilibrium. This equilibrium definition includes \((AC)\) in the household problems.

**Definition 2** \(\left( (x^h, \tilde{\gamma}^h)_{h \in \mathcal{H}}, p \right) \) is a financial equilibrium if

1. \( \forall h \in \mathcal{H}, \) given \( p, (\tilde{\gamma}^{h'})_{h' \neq h} \),
   \( (x^h, \tilde{\gamma}^h) \) is an optimal solution to the household problem \((HP)\), where (i) \( \tilde{\gamma}^h \) replaces \( \gamma^h \) in both the budget constraints and \((CL)\) and (ii) \((AC)\) is added as constraint 3.

2. markets clear:
   \[
   \sum_{h \in \mathcal{H}} \left( x^h_l(s) - e^h_l(s) \right) = 0 \quad \forall (l,s).
   \]

#### 2.4.1 Comparing a financial equilibrium with an unconstrained financial equilibrium

I assume for simplicity that trivial contracts are always accepted by households. This does not affect the results in any way, only the way that I describe equilibrium choices. Suppose that a household \(h\) makes a nontrivial contract proposals that is rejected. As contracts are simultaneously proposed, this household \(h\) is indifferent between making this nontrivial

\[7\text{The notation } M^{h'} \cdot \tilde{\Gamma}^{h'} (\tilde{\gamma}(\mathcal{h}, \mathcal{h'}), \tilde{\gamma}(h, h')) \text{ simply specifies that if } \tilde{\Gamma}^{h'} (\tilde{\gamma}(\mathcal{h}, \mathcal{h'}), \tilde{\gamma}(h, h')) = (\tilde{\gamma} (h', h' + 1), \ldots, \tilde{\gamma}(h', H)), \text{ then } M^{h'} \cdot \Gamma^{h'} (\tilde{\gamma}(\mathcal{h}, \mathcal{h'}), \tilde{\gamma}(h, h')) = \sum_{h'' > h'} \tilde{\gamma}(h', h''). \]
proposal and proposing the trivial contract. Thus, the set of financial equilibria is a subset of the set of unconstrained financial equilibria. Households only proposing contracts that are accepted do not have an incentive to deviate once this restriction is removed.

What I do in the sequel is provide conditions under which any financial equilibrium allocation is Pareto optimal. Under these conditions, I prove the existence of a financial equilibrium. The proof proceeds by construction beginning with the Pareto optimal allocation (see Theorem 4 in Subsection 3.3). Next, I show that the set of financial equilibria is allocation-equivalent to the set of unconstrained financial equilibria. To accomplish this, I first characterize the set of financial equilibria. I then show that there does not exist an unconstrained financial equilibrium allocation that lies outside this set (see Theorem 5 in Subsection 3.4). This justifies my earlier claim that the decision by households to make contract proposals subject to (AC) is an equilibrium choice.

3 "Complete" Markets

The following version of the "First Basic Welfare Theorem" motivates the analysis in this section as a connection is made between complete markets in GEI models and households' contract limits.

**Theorem 1** Under Assumptions 1-5, if \( \eta^h = H - 1 \) \( \forall h \in \mathcal{H} \), then any financial equilibrium allocation is Pareto optimal.

**Proof.** Suppose otherwise, that is, a financial equilibrium \( \left( (x^h, \tilde{\tau}^h)_{h \in \mathcal{H}}, p \right) \) exists with a suboptimal allocation. Using Assumption 3, the allocation is pairwise Pareto suboptimal for some pair \( (h, h') \), where \( h < h' \) (without loss of generality).\(^8\) As the commodity markets are Walrasian, then for any state \( s \in \mathcal{S} \), holding the total contract payoffs \( (\tau^h(s), \tau^{h'}(s))_{s \in \mathcal{S}} \) fixed, the allocation \( (x^h, x^{h'}) \) is Pareto optimal among all allocations that are feasible according to the fixed \( (\tau^h(s), \tau^{h'}(s))_{s \in \mathcal{S}} \). As the allocation \( (x^h, x^{h'}) \) is pairwise Pareto suboptimal, then \( \exists \alpha \in \mathbb{R}^{S+1} \setminus \{0\} \) such that the alternative allocation for the two households \( (\hat{x}^h, \hat{x}^{h'}) \), reached using nominal contracts \( (\tau^h + \alpha, \tau^{h'} - \alpha) \), strictly dominates the original equilibrium allocation: \( (u^h(\hat{x}^h), u^{h'}(\hat{x}^{h'})) \gg (u^h(x^h), u^{h'}(x^{h'})) \). This contradicts that the contract \( \tilde{\tau} (h, h') \) is an optimal solution to the household problem \( (HP) \) for household \( h \). This is because the households can never violate their contract limits and the alternative contract \( \tilde{\tau} (h, h') + \alpha \) is always accepted by \( h' \) and makes \( h \) strictly better off. \( \blacksquare \)

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\(^8\)See Gale (2000), citing previous works by Feldman (1973) and Goldman and Starr (1982), for the specific conditions under which Pareto optimality is equivalent to pairwise Pareto optimality.
Motivated by the above result, I examine the implications of bilateral bargaining for the Pareto optimality of the equilibrium allocations. I define a branch as the set of households that are linked through nontrivial contract offers (a formal definition is provided in the following subsection). The main result of this paper states that if the contract limits allow all households to belong to one branch, then any financial equilibrium allocation is Pareto optimal. This is a statement of the "First Basic Welfare Theorem" for this model of bilateral bargaining.

**Theorem 2** Under Assumptions 1-5, if all households can be linked through nontrivial contract offers, then any financial equilibrium allocation is Pareto optimal.

This section develops Theorem 2. Subsection 3.1 analyzes the first order conditions with respect to the households’ contract choices \( \tilde{\gamma}^h \). Subsection 3.2 proves that any financial equilibrium allocation is Pareto optimal under the conditions of Theorem 2.

### 3.1 Branches

In the household problem \((HP)\), let \( \lambda^h \in \mathbb{R}^{S+1} \) be the Lagrange multiplier for the budget constraints of household \( h \) and let \( \mu^{h,h'} \in \mathbb{R} \) be the Lagrange multiplier for \((AC)\) of a proposal from household \( h \) to household \( h' \). Recalling \((AC)\) from Subsection 2.3 and the fact that the best response \( \tilde{\gamma}^{h'} (\tilde{\gamma} (\emptyset, h', \tilde{\gamma} (h, h')) \) is one of the maximizers of the household problem \((HP)\), then I define

\[
V (\tilde{\gamma} (\emptyset, h', \tilde{\gamma} (h, h'))) = \max_{\tilde{\gamma}^{h'}} \left\{ \sigma^{h'} \left[ M^{h'} \cdot \tilde{\gamma}^{h'} - N^{h'} \cdot \left( \tilde{\gamma} (\emptyset, h') \right) \right] \right\},
\]

with maximizer \( \tilde{\gamma}^{h'} = \tilde{\gamma}^{h'} (\tilde{\gamma} (\emptyset, h', \tilde{\gamma} (h, h'))) \). From the Envelope Theorem, the derivative of \((AC)\) with respect to \( \tilde{\gamma} (h, h') \) is given by:

\[
\frac{\partial V (\tilde{\gamma} (\emptyset, h', \tilde{\gamma} (h, h')))}{\partial \tilde{\gamma} (h, h')} = \frac{\partial u^{h'} (\sigma^{h'} (\tilde{\gamma}^{h'}))}{\partial \tilde{\gamma} (h, h')} \bigg|_{\tilde{\gamma}^{h'} = \tilde{\gamma}^{h'}}
\]

where \( \tau^{h'} = \sum_{h'' > h', h'' \neq h'} \tilde{\gamma}^{h'} (h', h'') - \sum_{h'' < h', h'' \neq h'} \tilde{\gamma} (h'', h') - \tilde{\gamma} (h, h') \).

The first order condition for household \( h \) with respect to the nontrivial contract offer \( \tilde{\gamma} (h, h') \in \mathbb{R}^{S+1} \backslash \{0\} \) is:

\[
\lambda^h + \mu^{h,h'} \cdot Du^{h'} (x^{h'}) \cdot \frac{\partial \sigma^{h'} (\tau^{h'})}{\partial \tilde{\gamma} (h, h')} = 0_{1 \times (S+1)}.
\]
The static household problem \((SHP)\), in which the financial payoffs are held fixed at \((\tau^h)_{h \in \mathcal{H}}\), is given by:
\[
\max_{x^h \in X^h} u^h(x^h) \\
\text{subj. to } P(e^h - x^h) + \tau^h \geq 0.
\]

The function \(x^{h'} = \sigma^{h'}(\tau^{h'})\) is an implicit function of the system of equations characterizing a solution of \((SHP)\):
\[
F(x^{h'}, \lambda^{h'}; \tau^{h'}) = \left( \begin{array}{c} Du^{h'}(x^{h'}) - \lambda^{h'} P \\ P(e^{h'} - x^{h'}) + \tau^{h'} \end{array} \right) = 0_{(G+S+1) \times 1}.
\]

Using the Implicit Function Theorem:
\[
\frac{\partial \sigma^{h'}(\tau^{h'})}{\partial \tau^{h'}} = - \left[ I_G \mid 0_{G \times S+1} \right] \cdot [D_{x,\lambda}F(\cdot)]^{-1} \cdot [D_{\tau}F(\cdot)]
\]
\[
= - \left[ I_G \mid 0_{G \times S+1} \right] \cdot \left[ D^2 u^{h'}(x^{h'}) - P^T \begin{array}{c} -P \\ -P \end{array} \right]^{-1} \cdot \left[ 0_{G \times S+1} \mid I_{S+1} \right].
\]

Using the properties of blockwise matrix inversion, the product reduces to:
\[
\frac{\partial \sigma^{h'}(\tau^{h'})}{\partial \tau^{h'}} = - \left( D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \left[ -P \left( D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right]^{-1}.
\]

From the first order conditions for household \(h'\) with respect to \(x^{h'}, Du^{h'}(x^{h'}) = \lambda^{h'} P\), Eq. 1 reduces to:
\[
\lambda^h + \mu^{h,h'} \cdot \lambda^{h'} \cdot \left[ -P \left( D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right] \left[ -P \left( D^2 u^{h'}(x^{h'}) \right)^{-1} P^T \right]^{-1} \cdot \frac{\partial \tau^{h'}}{\partial \gamma(h, h')} = 0_{1 \times (S+1)}
\]

or simply
\[
\lambda^h - \mu^{h,h'} \cdot \lambda^{h'} = 0. \tag{2}
\]

From the first order conditions with respect to \(x^h, \lambda^h \gg 0 \ \forall h \in \mathcal{H}\). Then Eq. 2 implies that \(\mu^{h,h'} > 0\) for any household \(h\) making a nontrivial contract offer to \(h'\).

Suppose that household \(h\) offers a nontrivial contract to household \(h'\), but not \(h''\). If household \(h'\) offers a nontrivial contract to \(h''\), the pair \((h, h'')\) are still connected using two iterations of Eq. 2:
\[
\lambda^h = \mu^{h,h'} \cdot \lambda^{h'} = \mu^{h,h'} \cdot \left( \mu^{h',h''} \cdot \lambda^{h''} \right). \tag{3}
\]

Further, suppose that both \(h\) and \(h''\) offer a nontrivial contract to household \(h'\). The pair
(h, h') are still connected using two iterations of Eq. 2:

\[
\lambda^h = \mu^{h,h'} \cdot \lambda^{h'} = \mu^{h,h'} \left( \frac{\lambda^{h''}}{\mu^{h'',h'}} \right).
\]

If all households h' > 1 have a nontrivial contract connection with h = 1, or with any household that has a nontrivial contract connection with h = 1, or with any household that has a nontrivial contract connection with a household that has a nontrivial contract connection with h = 1, ..., and so forth, then Eqs. 3 and 4 imply that

\[ \forall h' \neq h : \lambda^h = \kappa^{h'} \cdot \lambda^{h'} \quad \text{for some } \kappa^{h'} \in \mathbb{R}_{++}. \]

Therefore, the resulting financial equilibrium allocation is Pareto optimal.

Define the branch of households originating with household h = 1 and connecting all households h' : \( \lambda^h = \kappa^{h'} \cdot \lambda^{h'} \) for some \( \kappa^{h'} \in \mathbb{R}_{++} \) as the set \( \mathcal{H}_1^* \subseteq \mathcal{H} \). These households are usually connected through an iterated linking of nontrivial contracts, though it is possible that \( \lambda^h = \kappa^{h'} \cdot \lambda^{h'} \) for some \( \kappa^{h'} \in \mathbb{R}_{++} \) even without such a contract connection.

Suppose that household \( h'' \notin \mathcal{H}_1^* \). This requires that for any \( \kappa^{h''} \in \mathbb{R}_{++}, \lambda^1 \neq \kappa^{h''} \lambda^{h''} \). This implies that \( h'' \) does not have a contract connection with any household \( h' \in \mathcal{H}_1^* \). Thus, a second branch can be defined where household \( h_2^* = \min \{ \mathcal{H} \setminus \mathcal{H}_1^* \} \) is the household that can make contract offers to all of the remaining households \( h' \in \mathcal{H} \setminus \mathcal{H}_1^* \). Define the branch of households originating with household \( h_2^* \) and connecting all households h' : \( \lambda^{h_2^*} = \kappa^{h'} \cdot \lambda^{h'} \) for some \( \kappa^{h'} \in \mathbb{R}_{++} \) as the set \( \mathcal{H}_2^* \subseteq \mathcal{H} \setminus \mathcal{H}_1^* \). By induction, define branches \( i = 1, ..., I \) so that \( \mathcal{H} = \mathcal{H}_1^* \cup ... \cup \mathcal{H}_I^* \). For simplicity, define \( h_i^* = 1 \) and \( h_i^* (h) = \min \{ \mathcal{H} \setminus (\bigcup_{k<i} \mathcal{H}_k^*) \} \) for \( i = 2, ..., I \). By definition, \( \forall h' \in \mathcal{H}_i^*, \lambda^{h_2^*} = \kappa^{h'} \cdot \lambda^{h'} \) for some \( \kappa^{h'} \in \mathbb{R}_{++} \).

### 3.2 First Basic Welfare Theorem

The branches themselves, namely the set of households that comprise each branch, are endogenously determined. Changes in the parameters \( (e^h, u^h)_{h \in \mathcal{H}} \) can lead to the formation of different branches. Yet, the number of branches, \( I \), is constant across changes in \( (e^h, u^h)_{h \in \mathcal{H}} \). The number of branches, \( I \), is an exogenous function of the contract limits \( (\eta^h)_{h \in \mathcal{H}} \) as these parameters govern the number of links that can be formed. Theorem 3 verifies that the number of branches is exogenous. In particular, this requires showing that if it’s possible for a branch to contain all remaining households \( h' \in \mathcal{H} \setminus (\bigcup_{k<i} \mathcal{H}_k^*) \), then it is also optimal for household \( h_i^* \) to offer contracts so that the endogenous composition of branch \( i \) is \( \mathcal{H}_i^* = \mathcal{H} \setminus (\bigcup_{k<i} \mathcal{H}_k^*) \).
**Definition 3** The set of households $\mathcal{H}^+$ is achievable if for any allocation $(y^{h'})_{h' \in \mathcal{H}^+}$ such that:

1. $\sum_{h' \in \mathcal{H}^+} (y^{h'}_l(s) - e^{h'}_l(s)) = 0 \ \forall (l, s)$ and
2. for $h = \min \mathcal{H}^+$, then $\forall h' \in \mathcal{H}^+ \setminus \{h\}$, $Du^h (y^h) = \kappa^{h'} Du^{h'} (y^{h'})$ for some $\kappa^{h'} \in \mathbb{R}_{++}$, then the contracts required to achieve the allocation $(y^{h'})_{h' \in \mathcal{H}^+}$ satisfy $(CL) \ \forall h' \in \mathcal{H}^+$.

**Theorem 3** For any $i$, if a branch linking all remaining households $h' \in \mathcal{H} \setminus (\bigcup_{k<i} \mathcal{H}^+_k)$ is achievable, then household $h_i^* = \min \{\mathcal{H} \setminus (\bigcup_{k<i} \mathcal{H}^+_k)\}$ finds it optimal to offer contracts such that $\mathcal{H}^*_i = \mathcal{H} \setminus (\bigcup_{k<i} \mathcal{H}^*_k)$.

**Proof.** See Section 7.1. ■

Under Assumption 4, which states that $\eta^h \geq 1 \ \forall h \in \mathcal{H}$ and $\sum_{h \in \mathcal{H}} \eta^h$ is even, the smallest branch contains two households. All households are permitted to offer or accept at least one nontrivial contract. Failure to do so implies that a household $h$ is already a part of a branch with another household $h' : \lambda^{h'} = \kappa^h \lambda^h$ for some $\kappa^h \in \mathbb{R}_{++}$. So, the number of branches is bounded: $I \leq \frac{\mathcal{H}}{2}$.

Given Theorem 3 and the analysis in Subsection 3.1, I can restate Theorem 2, the main result of this paper, as a Corollary.

**Corollary 1** Under Assumptions 1-5, if a branch linking all households $\mathcal{H}$ is achievable, then any financial equilibrium allocation is Pareto optimal.

### 3.3 Existence

The following theorem shows that under the same conditions used for Corollary 1, the existence of a financial equilibrium is guaranteed. The proof proceeds by construction, beginning with a Pareto optimal allocation.

**Theorem 4** Under Assumptions 1-5, if a branch linking all households $\mathcal{H}$ is achievable, then a financial equilibrium exists.

**Proof.** See Section 7.2. ■
3.4 Equilibria equivalence

The following result verifies the equivalence (in terms of equilibrium allocation) between the set of unconstrained financial equilibria and the set of financial equilibria. Thus, moving forward, I state results for the set of financial equilibria.

**Theorem 5** The set of unconstrained financial equilibria is allocation-equivalent to the set of financial equilibria.

**Proof.** See Section 7.3. ■

4 "Incomplete" Markets

From the previous section, the number of branches, $I$, is determined exogenously. The composition of the branches, specifically the sets $\mathcal{H}_1^*, ..., \mathcal{H}_I^*$, are endogenous. For the remainder of the section, I hold fixed the sets $\mathcal{H}_1^*, ..., \mathcal{H}_I^*$. The results that I obtain for any one composition of the branches will hold for any of the other possible compositions.

I need to introduce some additional notation and the mathematical tools necessary to prove the generic regularity of financial equilibria. As a corollary of the regularity result, I prove that if more than one branch exists, $I > 1$, then any financial equilibrium allocation will be generically Pareto suboptimal (Subsection 4.1). Moreover, if $1 < I \leq S+1$ and $L > 1$, then any financial equilibrium allocation will be generically constrained Pareto suboptimal (Subsection 4.2).

4.1 Regularity

I begin by normalizing the commodity prices: $p_L(s) = 1 \forall s \in S$.

We consider the household problem of household $h$. From the point of view of household $h$, the outside utility of any household $h' > h$ is the utility that household $h'$ could receive by rejecting the offer $\tilde{\gamma}(h, h')$ and then proposing a new vector of optimal contracts. Denote this outside utility as $\tilde{u}^{h'}(h)$, where:

$$\tilde{u}^{h'}(h) = u^{h'} \left\{ \alpha^{h'} \left[ M^{h'} \cdot \tilde{\Gamma}^{h'} (\gamma^{h'}, 0) - N^{h'} \cdot \left( \begin{array}{c} \tilde{\gamma}(\gamma^{h'}, h') \\ 0 \end{array} \right) \right] \right\}$$

as in (AC). In equilibrium, as we will see below, the optimal contract proposal $\tilde{\gamma}(h, h')$ will be such that $u^{h'}(x^{h'}) = \tilde{u}^{h'}(h)$. 

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From the point of view of a social planner, the outside utility of the same household $h'$ is the utility that household $h'$ can receive by rejecting all offers $(\tilde{\gamma}(h, h'))_{h < h'}$ and then proposing a new vector of optimal contracts. In my notation, this outside utility has value

$$\tilde{u}_{SP}^{h'} = u^{h'} \left\{ \sigma^{h'} \left[ M^{h'} \cdot \tilde{\Gamma}^{h'} \left( (\tilde{\gamma}(h, h''))_{h'' \neq h'}, 0 \right)_{h < h'}, (\tilde{\gamma}^h)_{h > h'} \right] \right\}.$$ 

In equilibrium, for any $h < h'$, $u^{h'}(x^{h'}) = \tilde{u}^h(h)$, so therefore $\tilde{u}_{SP}^{h'} = \tilde{u}^h(h) \; \forall h < h'$. I simply denote these identical outside utility values for household $h'$ as $\tilde{u}^h$. For future reference,

$$\tilde{u}^{h'} = u^{h'} \left\{ \sigma^{h'} \left[ M^{h'} \cdot \tilde{\Gamma}^{h'} (\tilde{\gamma}(\emptyset, \emptyset), 0) - N^{h'} \cdot \begin{pmatrix} \tilde{\gamma}(\emptyset, h') \\ 0 \end{pmatrix} \right] \right\}, \quad (5)$$

where $h$ can be any household $h < h'$.

### 4.1.1 Updating the household problem

In the problem $(HP)$ for household $h \in \mathcal{H}^+_t$, the values $(\tilde{u}^h)_{h \in \mathcal{H}^+_t : h > h}$ are variables for household $h$. Define $\mathcal{H}^+_h$ as the set of households $h''$ that will be connected via a series of nontrivial contracts to household $h'$ according to the contract proposals $\tilde{\Gamma}^{h'} (\tilde{\gamma}(\emptyset, \emptyset), 0)$ (the household $h'$ deviates and rejects the contract proposals it receives).

I need to be specific about how households make optimal deviations from the equilibrium path. For any set of households $\mathcal{H}''$ of size $H''$ and any household $h' \notin \mathcal{H}''$, define $\phi_{H'', h'} : \mathbb{R}^{H''} \to \mathbb{R}$ as the $C^2$ implicit function (the utility $u^{h'}(\hat{x}^{h'}) = \phi_{H'', h'} \left( (\tilde{u}^h)_{h \in \mathcal{H}''} \right)$) of the $C^1$ system of equations $\Phi^{h'}_h$ given below (containing first order conditions, Walras’ Law, and Acceptance conditions), with $n = G (H'' + 1)$ equations and $n$ unknowns $\hat{x} = (\hat{x}^h)_{h \in \mathcal{H}''}$:

$$\Phi^{h'}_h(\hat{x}; p) = \left\{ \begin{array}{ll}
\forall h \in \mathcal{H}'' \cup \{h'\}, \\
\forall l < L, \text{ and } \forall s \in S : & p_l(s) = \frac{D_{(L,s)}u^h(\hat{x}^h)}{D_{(L,s)}u^h(\hat{x}^h)} \\
\sum_{h \in \mathcal{H}'' \cup \{h'\}} \left[ \sum_{l < L} p_l(s) \left( \hat{x}^h_l(s) - e^h_l(s) \right) + \hat{x}^h_{L}(s) - e^h_{L}(s) \right] = 0 \\
\forall h \in \mathcal{H}'' \text{ and } \forall s > 0 : & \frac{D_{(L,s)}u^h(\hat{x}^h)}{D_{(L,s)}u^h(\hat{x}^h)} = \frac{D_{(L,s)}u^{h'}(\hat{x}^{h'})}{D_{(L,s)}u^{h'}(\hat{x}^{h'})} \\
\forall h \in \mathcal{H}'' : & u^h(\hat{x}^h) = \tilde{u}^h
\end{array} \right\}.$$ 

The system of equations $\Phi^{h'}_h$ does not contain market clearing conditions, because the choices
by households need not be equilibrium choices. They are only "off the equilibrium path" choices. When making either equilibrium or "off the equilibrium path" choices, households always take the commodity prices \( p \) as given.

In an economy with only one commodity, \( L = 1 \), the equations \( \Phi^*_{h^i}(\cdot) = 0 \) characterize the point on the utility possibilities frontier (for the subeconomy of households \( h \in \mathcal{H}^\prime \cup \{h^i\} \)) corresponding to \( u^h(\hat{x}^h) = \tilde{u}^h \forall h \in \mathcal{H}^\prime \). We know that the utility possibilities frontier is such that the implicit function \( u^{h^i} = \phi_{\mathcal{H}^\prime, h^i} \left((\tilde{u}^h)_{h \in \mathcal{H}^\prime}\right) \) is differentiably strictly decreasing and differentiably strictly concave. In similar fashion, Assumption 3 guarantees that for economies with more than one commodity, \( L > 1 \), the function \( \phi_{\mathcal{H}^\prime, h^i} \) is differentiably strictly decreasing and differentiably strictly concave.

To formally account for the situation in which the values \( (\tilde{u}^{h^i})_{h^i \in \mathcal{H}^*; h^i > h} \) are variables for household \( h \in \mathcal{H}_i^* \), the household problem \( (HP) \) is updated as:

\[
\begin{align*}
\max_{x^h, z^h, (\tilde{u}^{h^i})_{h^i \in \mathcal{H}^*; h^i > h}} & \quad u^h(x^h) \\
\text{subj. to} & \quad P(e^h - x^h) + \sum_{h^i \neq h} \tilde{\gamma}(h, h^i) - \sum_{h^i < h} \tilde{\gamma}(h^i, h) \geq 0. \\
\end{align*}
\]

1. Constraint \( (CL) \) holds.
2. Constraint \( (AC) \) holds for all \( h^i > h \).
3. \( \forall h^i \in \mathcal{H}_i^* : h^i > h \),
4. \( \tilde{u}^{h^i} = \phi_{\mathcal{H}^\prime, h^i} \left((\tilde{u}^{h^i})_{h^i \in \mathcal{H}^\prime} \cap \mathcal{H}_i^* ; (\tilde{u}^{h^i})_{h^i \in \mathcal{H}^\prime} \setminus \mathcal{H}_i^* \right) \).

The constraint \( \tilde{u}^{h^i} = \phi_{\mathcal{H}^\prime, h^i} \left((\tilde{u}^{h^i})_{h^i \in \mathcal{H}^\prime} \cap \mathcal{H}_i^* ; (\tilde{u}^{h^i})_{h^i \in \mathcal{H}^\prime} \setminus \mathcal{H}_i^* \right) \geq 0 \) is strictly increasing and strictly convex in the variables \( (\tilde{u}^{h^i})_{h^i \in \mathcal{H}^\prime} \cap \mathcal{H}_i^* \). The maximizers \( (\tilde{u}^{h^i})_{h^i \in \mathcal{H}^\prime} \cap \mathcal{H}_i^* \) of the household problem Eq. 6 are not interior solutions, but are determined by Eq. 5. Then, \( \tilde{u}^{h^i} \) is determined from constraint 4. Other possible maximizers to the problem Eq. 6 are the variables such that for some \( k \in \mathcal{H}_h^+ \cap \mathcal{H}_i^* \), the values \( (\tilde{u}^{h^i}, (\tilde{u}^{h^i})_{h^i \in \mathcal{H}^\prime \setminus \mathcal{H}_i^* \setminus \{k\}} \) are determined by the boundary conditions and \( \tilde{u}^k \) is implicitly determined from the constraint 4.

The analysis indicates that household \( h \) has a discrete choice in the problem Eq. 6. This discrete choice permits real multiplicity of financial equilibria. This multiplicity is not ignored in the sequel, but the results are presented under the assumption that the values \( (\tilde{u}^h)_{h \in \mathcal{H} \setminus \mathcal{H}_i^*} \) are held fixed.

These discrete choices do no affect the regularity of financial equilibria. If a household is indifferent between these discrete choices then real multiplicity arises. This multiplicity is not
problematic as the consumption and utility remain unchanged for the indifferent household. If a household is not indifferent, then it is not indifferent for all economies within an open neighborhood of the specified economy.

### 4.1.2 Equilibrium equations

With the values \((\bar{u}^h)_{h \in \mathcal{H}\setminus\{h^*_1, \ldots, h^*_i\}}\) held fixed, the order of nontrivial contracts does not have any real effects. For instance, the financial equilibrium allocation in Figure 1(a) is equivalent to the financial equilibrium allocation in Figure 1(b).

Thus, I can ignore the order of nontrivial contracts and use an equivalent formulation of the contract variables.

Consider branch \(i\). As all households \(h \in \mathcal{H}_i^*\setminus\{h^*_i\}\) are connected to household \(h^*_i\) and have fixed values of \((\tilde{u}^h)_{h \in \mathcal{H}_i^*\setminus\{h^*_i\}}\), it is innocuous (in terms of the real variables) to replace the original contracts \((\tilde{h}^h)_{h \in \mathcal{H}_i^*\setminus\{h^*_i\}}\) with the alternative contracts \((\hat{h}^h)_{h \in \mathcal{H}_i^*\setminus\{h^*_i\}}\). The alternative contracts \((\hat{h}^h)_{h \in \mathcal{H}_i^*}\) are defined relative to the original contracts \((\tilde{h}^h)_{h \in \mathcal{H}_i^*\setminus\{h^*_i\}}\) so that the nominal transfers \((\tau^h)_{h \in \mathcal{H}_i^*\setminus\{h^*_i\}}\) remain unchanged:

\[
\tau^h = -\hat{\gamma}(h^*_i, h) \quad \forall h \in \mathcal{H}_i^*\setminus\{h^*_i\}
\]

The households \(h \in \mathcal{H}_i^*\setminus\{h^*_i\}\) have a static household problem given by \((SHP)\) in Subsection 3.1.

Rather than consider the household problem for all households \(h \in \mathcal{H}_i^*\), where the transfers \((\tau^h)_{h \in \mathcal{H}_i^*}\) are variables, we can equivalently characterize equilibria by eliminating the variables \((\tau^h)_{h \in \mathcal{H}_i^*}\) from the budget constraints of all households \(h \in \mathcal{H}_i^*\setminus\{h^*_i\}\) and only considering an updated household problem for \(h^*_i\). This updated household problem for \(h^*_i\)

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9 The Addendum provides a 3-household example in which a continuum of financial equilibria exist under the conditions (i) \(L = 1\) and (ii) \(h = 2, 3\) have identical homothetic utility functions. Yet, in each equilibrium, \(h = 1\) has the same consumption. Further, when \(L > 1\), such a continuum is impossible provided that households have heterogeneous preferences.
The Kuhn-Tucker conditions are given by:

$$\max_{(x^h)_{h \in \mathcal{H}_i^*}} u^h(x^h)$$

subject to

1. $P(e_i^h - x_i^h) + \sum_{h \in \mathcal{H}_i^* \setminus \{h_i^*\}} P(e^h - x^h) \geq 0.$ (7)

2. $\forall h \in \mathcal{H}_i^* \setminus \{h_i^*\}: u^h(x^h) \geq \tilde{u}^h.$

The Kuhn-Tucker conditions are necessary and sufficient for an optimal solution of Eq. 7. The Kuhn-Tucker conditions are given by:

1. first order conditions, consumption (FOC$_x$) : $Du^h(x^h) - \lambda^h P = 0 \ \forall h \in \mathcal{H}_i^*,$ where (a) $\lambda^h_i$ is the Lagrange multiplier with respect to the equation $P(e_i^h - x_i^h) + \sum_{h' \in \mathcal{H}_i^* \setminus \{h_i^*\}} P(e_{i'}^h - x_i^h) \geq 0,$ (b) $(\mu^h)_{h \in \mathcal{H}_i^* \setminus \{h_i^*\}}$ are the Lagrange multipliers with respect to the equations $(u^h(x^h) \geq \tilde{u}^h)_{h \in \mathcal{H}_i^* \setminus \{h_i^*\}},$ and (c) $\lambda^h = \frac{\lambda^h_i}{\mu^h} \ \forall h \in \mathcal{H}_i^* \setminus \{h_i^*\}.$

2. budget constraints (BC) : $P(e_i^h - x_i^h) + \tau^h = 0 \ \forall h \in \mathcal{H}_i^*,$ where $\tau_i^{h'} = -\tilde{\gamma}(h_i^*, h') \ \forall h' \in \mathcal{H}_i^* \setminus \{h_i^*\}$ and $\tau_i^{h_i^*} = \sum_{h' \in \mathcal{H}_i^* \setminus \{h_i^*\}} \tilde{\gamma}(h_i^*, h').$

3. first order conditions, contracts (FOC$_\tilde{\gamma}$) : $\mu^h \lambda^h = \lambda_i^{h_i^*} \ \forall h \in \mathcal{H}_i^* \setminus \{h_i^*\}.$

4. Acceptance conditions (AC) : $u^h(\sigma^h(-\tilde{\gamma}(h_i^*, h))) = \tilde{u}^h \ \forall h \in \mathcal{H}_i^* \setminus \{h_i^*\}.$

Now considering all branches $i = 1, \ldots, I$, define the set of variables as

$$\xi = \left( (x^h, \lambda^h)_{h \in \mathcal{H}}, (\tilde{\gamma}^h)_{h = h_i^*, \ldots, h_i^*}, (\mu^h)_{h \in \mathcal{H} \setminus \{h_i^*, \ldots, h_i^*\}}, \tilde{\mu} \right).$$

From the first order conditions with respect to consumption, Assumption 3, and Eq. 3, the Lagrange multipliers $\lambda^h \in \mathbb{R}_{++}^{S+1}$ and $\mu^h \in \mathbb{R}_{++}$ (open sets). Then $\xi \in \Xi = \times_{h \in \mathcal{H}} (X^h \times \mathbb{R}_{++}^{S+1}) \times \mathbb{R}^{(H-I)(S+1)} \times \mathbb{R}_{++}^{H-I} \times \mathbb{R}_{++}^{G-(S+1)},$ an open set. Define the set of parameters as $\theta = (e^h, u^h)_{h \in \mathcal{H}}.$ The parameters belong to the set $\Theta = \mathcal{E} \times \mathcal{U},$ also an open set.

The equilibrium manifold $\Phi : \Xi \times \Theta \to \mathbb{R}^n$ is defined by $n$ equations (where $n = H(G + S + 1) + (S + 2)(H - I) + G - (S + 1)$) so that $\xi \in \Xi$ is a financial equilibrium iff $\Phi(\xi, \theta) = 0,$ where $\Phi = \begin{pmatrix} FOC_x \\ BC \\ FOC_{\tilde{\gamma}} \\ AC \\ MC_x \end{pmatrix}$ and the market clearing conditions (MC$_x$) are...
Define the equilibrium projection as:

\[
\left( \sum_{h \in \mathcal{H}} (e^h_{\mathcal{L}}(s) - x^h_{\mathcal{L}}(s)) \right)_{s \in S} \, .
\]

The mapping \( \pi \) is proper iff \( \pi \) is \( C^0 \) and for compact \( \Theta' \subset \Theta \), \( \pi^{-1}(\Theta') \) is also compact.

### 4.1.3 Regularity results

**Theorem 6** Under Assumptions 1-5, the set of regular values of \( \pi \) is a generic subset of \( \Theta \). Specifically, all financial equilibria \( \pi^{-1}(\theta) \) are regular for all \((u^h)_{h \in \mathcal{H}} \in \mathcal{U}\) and for all \((e^h)_{h \in \mathcal{H}} \) in a generic subset of \( \mathcal{E} \).

**Proof.** To prove Theorem 6, it suffices to prove that (a) \( \pi \) is proper and (b) \( D_{\xi,e} \Phi (\xi, \theta) \) has full row rank \( n \) whenever \( \Phi (\xi, \theta) = 0 \). The proof of part (a) is standard and the details can be found in Villanacci et al. (2002). The proof of part (b) is contained in Section 7.4.

As a Corollary to Theorem 6, I obtain the following useful facts. The first will be used in the proof of Theorem 7. The second fact implies that any financial equilibrium allocation is Pareto suboptimal over a generic subset of household endowments \( \mathcal{E} \), provided that more than one branch is formed.

**Corollary 2** Under Assumptions 1-5, over a generic subset of household endowments \( \mathcal{E} \), the following facts are obtained:

(i) \( \exists l \) without loss of generality \( l = 1 \), so that \( \sum_{h \in \mathcal{H}'} (x^h_1(s) - e^h_1(s)) \neq 0 \) for any subset \( \mathcal{H}' \subset \mathcal{H} \) with \( \mathcal{H}' \neq \mathcal{H} \), for any financial equilibrium, and for all states \( s \in S \).

(ii) if \( I > 1 \), then \( \exists h, h' \) so that \( \frac{\lambda^{(1)}}{\lambda^{(0)}}(s) \neq \frac{\lambda'^{(1)}}{\lambda'^{(0)}}(s) \) for any financial equilibrium.

**Proof.** See Section 7.5.

### 4.2 Generic constrained Pareto suboptimality

The previous subsection demonstrated that any financial equilibrium allocation is generically Pareto suboptimal. The comparison between an equilibrium allocation and a Pareto optimal allocation seems unfair as a Pareto optimal allocation implicitly allows for transfers between any pair of households. Given the contract limits, an equilibrium allocation can only be achieved through a limited number of bilateral contracts. I define the property of constrained Pareto optimality to incorporate this restriction.

For any financial equilibrium, the number of branches is \( I \). By the definition of a branch, a nominal transfer can be made between any two households in a branch as these households are linked through a series of contracts.

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\(^{10}\)The notation \( x_{\setminus \mathcal{L}}(s) \) denotes the column vector containing the elements \( l \neq \mathcal{L} : (x_1(s), \ldots, x_{L-1}(s))^T \).
Definition 4 With respect to the $I$ branches, the vector of nominal transfers $(\tau^h)_{h \in \mathcal{H}} \in \mathbb{R}^{H(S+1)}$ is $I$-feasible if $\sum_{h \in \mathcal{H}} \tau^h = 0 \, \forall i = 1, \ldots, I$.

Definition 5 An allocation $(x^h)_{h \in \mathcal{H}}$ is constrained Pareto optimal (with respect to the $I$ branches) if there does not exist a vector of $I$-feasible transfers $(\tau^h)_{h \in \mathcal{H}} \in \mathbb{R}^{H(S+1)}$ such that the allocation $(\hat{x}^h)_{h \in \mathcal{H}}$ Pareto dominates $(x^h)_{h \in \mathcal{H}}$, where $(\hat{x}^h)_{h \in \mathcal{H}}$ are equilibrium consumption choices of the static household problem (SHP) (given in Subsection 3.1).

The following result proves that the bargaining framework under which households enter into branches is not sufficient to ensure that the financial equilibrium allocations are constrained Pareto optimal. That is, a planner can intervene with nominal transfers only between households in the same branch and still make all households strictly better off. The result requires that multiple physical commodities are traded in each state.

Assumption 6 $L > 1$.

Theorem 7 Under Assumptions 1-6, over a generic subset of household endowments and utility functions $\mathcal{E} \times \mathcal{U}$, if $1 < I \leq S + 1$, then any financial equilibrium allocation is constrained Pareto suboptimal.

Proof. See Section 7.6. ■

5 Asymmetric Information

This section analyzes the effects of asymmetric information. The theory of asymmetric information is discussed in Subsection 5.1. Basically, a household making contract proposals cannot distinguish among the households that it is proposing contracts to. Thus, it is possible that a contract intended for a certain household is actually accepted by a different household. An example of this nature is considered in Subsection 5.2. The key issue is whether the incentive compatibility constraints are satisfied at the original financial equilibrium. If they are not satisfied, then the contract proposals in the original financial equilibrium are not able to separate the households, so the equilibrium under asymmetric information must be different.

5.1 Theoretical setup

As in Section 2, the parameters $(\eta^h)_{h \in \mathcal{H}}$ and household labeling are common knowledge. Now, however, any given household $h$ cannot observe the household parameters $(e^h, u^h)$.
for $h' > h$. The household $h$ does observe the distribution $\{(e^{h'}, u^{h'})_{h'>h}\}$, but not which parameters belong with which household.

As a consequence, the contract proposals $\tilde{\gamma}^h = (\tilde{\gamma} (h, h'))_{h'>h}$ are not household specific, whereby $\tilde{\gamma} (h, h')$ is for household $h'$ and only $h'$, $\tilde{\gamma} (h, h'')$ is for household $h''$ and only $h''$, and so forth. Rather, the contract proposals are publicly observed offers and any proposal can be accepted by any household $h' > h$.

Compared to the timing of actions in Subsection 2.1, a household $h$ now has the following information when it is called upon to make its contract proposals $\tilde{\gamma}^h$: the commodity prices $p$, the contract proposals $(\tilde{\gamma}^{h'})_{h'\neq h}$, the household parameters $(e^{h'}, u^{h'})_{h'<h}$, the distribution of household parameters $\{(e^{h'}, u^{h'})_{h'>h}\}$, the constraints $(\eta^h)_{h \in H}$, and the ordering of households. The rest of the timing and the definition of a financial equilibrium remain the same as in Section 2.

5.2 Example 1

Consider an economy with three households $(H = 3)$, only one state of uncertainty in the final period $(S = 1)$, and two commodities traded in each state $(L = 2)$. Household $h = 1$ has a contract limit of $\eta^1 = 2$, while households $h = 2, 3$ have contract limits of $\eta^2 = \eta^3 = 1$. From Theorem 2, with complete information, the First Basic Welfare Theorem holds and all equilibrium allocations are Pareto optimal. The economy can be depicted in Figure 2.

![Diagram](image)

Figure 2: Example 1

The utility functions for all households $h \in H$ are of the Cobb-Douglas form:

$$u^h(x^h) = \alpha^h(0) \log (x_1^h(0)) + (1 - \alpha^h(0)) \log (x_2^h(0)) +$$

$$\beta^h \{\alpha^h(1) \log (x_1^h(1)) + (1 - \alpha^h(1)) \log (x_2^h(1))\}$$
The utility parameters are given in Table I.

<table>
<thead>
<tr>
<th></th>
<th>$\beta$</th>
<th>$\alpha(0)$</th>
<th>$\alpha(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3.5</td>
<td>0.228</td>
<td>0.567</td>
</tr>
<tr>
<td>2</td>
<td>0.49</td>
<td>0.25</td>
<td>0.824</td>
</tr>
<tr>
<td>3</td>
<td>0.31</td>
<td>0.444</td>
<td>0.5</td>
</tr>
</tbody>
</table>

Table I: Utility parameters

The household endowments are given in Table II.

<table>
<thead>
<tr>
<th></th>
<th>$e$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(5, 6, 3, 7)</td>
</tr>
<tr>
<td>2</td>
<td>(4, 6, 7, 3)</td>
</tr>
<tr>
<td>3</td>
<td>(8, 5, 3, 6)</td>
</tr>
</tbody>
</table>

Table II: Endowments

The unique financial equilibrium in the model with complete information can be calculated and is given in Table III.\(^{11}\)

<table>
<thead>
<tr>
<th></th>
<th>$p(0)$</th>
<th>$p(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma(1, 2)$</td>
<td>$(-1.569, 2.322)$</td>
<td>$(-1.569, 2.122)$</td>
</tr>
<tr>
<td>$\gamma(1, 3)$</td>
<td>$1.732, 2.928, 7.459, 11.412$</td>
<td>$8.637$</td>
</tr>
<tr>
<td>$\gamma(2, 3)$</td>
<td>$5.176, 7.765, 4.132, 1.771$</td>
<td>$2.569$</td>
</tr>
<tr>
<td>$\gamma(3, 1)$</td>
<td>$10.092, 6.307, 1.409, 2.818$</td>
<td>$2.262$</td>
</tr>
</tbody>
</table>

Table III: Financial equilibrium

With complete information, household $h = 1$ can make the offer $\gamma(1, 2)$ to only household $h = 2$, who in turn accepts the offer. Likewise, $h = 1$ can make the offer $\gamma(1, 3)$ to only household $h = 3$, who in turn accepts the offer. With asymmetric information, this is no longer possible. Consider what would happen if $h = 1$ were to make the same two offers as in Table III. Both households $h = 2, 3$ would elect to accept contract $\gamma(1, 3)$ as it dominates $\gamma(1, 2)$ (recall that payoffs for $h = 2, 3$ are $-\gamma(1, \cdot)$).

There are two options available to household $h = 1$ in the model with asymmetric information:

1. Offer a pooling contract $\gamma^* \in \mathbb{R}^2$ that is accepted by both households (this could include a trivial contract).

\(^{11}\)Uniqueness is guaranteed in this particular example, because in equilibrium both (i) $u^h(\sigma^h(0)) = u^h(e^h)$ for $h = 2, 3$, and (ii) $\phi_{h', h}(u^h(e^{h'})) = u^h(e^h)$ for $h, h' \in \{2, 3\}$ with $h \neq h'$. This allows $h = 1$ to make offers that only provide the utility of endowment to $h = 2, 3$. 

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2. Offer a menu of separating contracts \((\hat{\gamma}(2), \hat{\gamma}(3)) \in \mathbb{R}^2\) such that only \(h = 2\) accepts the contract \(\hat{\gamma}(2)\) and only \(h = 3\) accepts the contract \(\hat{\gamma}(3)\). The contract \(\hat{\gamma}(3) = \hat{\gamma}(1,3)\) can continue to be offered, but a new contract \(\hat{\gamma}(2)\) must be written so that both (i) \(h = 2\) accepts \(\hat{\gamma}(2)\) and not \(\hat{\gamma}(3)\) and (ii) \(h = 3\) continues to accept \(\hat{\gamma}(3)\).

**Pooling** If household \(h = 1\) elects to offer a pooling contract, then the optimal pooling contract \(\gamma^*\) is determined from the following maximization problem:

\[
\begin{align*}
\max_{\gamma^* \in \mathbb{R}^2} & \quad u^l(x^1) \\
\text{subj. to} & \quad 1. \quad u^2(x^2) \geq u^2(e^2) \quad (AC), \\
& \quad 2. \quad u^3(x^3) \geq u^3(e^3)
\end{align*}
\]

(8)

where the household consumption choices (as the utility functions are Cobb-Douglas) are given by

\[
x^1 = \left( \begin{array}{c}
\alpha^1(0) \frac{p(0)e^1(0) + 2\gamma_0}{p(0)e^1(0) + 2\gamma_0^*} \\
(1 - \alpha^1(0)) \frac{p(0)e^1(0) + 2\gamma_0}{p(0)e^1(0) + 2\gamma_0^*} \\
\alpha^1(1) \frac{p(1)e^1(1) + 2\gamma_1}{p(1)e^1(1) + 2\gamma_1^*} \\
(1 - \alpha^1(1)) \frac{p(1)e^1(1) + 2\gamma_1}{p(1)e^1(1) + 2\gamma_1^*}
\end{array} \right)
\]

\[
x^h = \left( \begin{array}{c}
\alpha^h(0) \frac{p(0)e^h(0) - \gamma_0^*}{p(0)e^h(0) - \gamma_0^*} \\
(1 - \alpha^h(0)) \frac{p(0)e^h(0) - \gamma_0^*}{p(0)e^h(0) - \gamma_0^*} \\
\alpha^h(1) \frac{p(1)e^h(1) - \gamma_1^*}{p(1)e^h(1) - \gamma_1^*} \\
(1 - \alpha^h(1)) \frac{p(1)e^h(1) - \gamma_1^*}{p(1)e^h(1) - \gamma_1^*}
\end{array} \right)
\]

for \(h = 2, 3\)

and the conditions \((AC)\) are correct as written given the facts from the footnote preceding Table III.

It is not possible for the contract \(\gamma^*\) to be such that \((AC)\) binds for both households. If the complete information contract offers are made, household \(h = 2\) does not accept the offer \(\hat{\gamma}(1,2)\), electing instead to accept \(\hat{\gamma}(1,2)\). Thus, the optimal pooling contract will be such that \((AC)\) does not bind for \(h = 2\), but does bind for \(h = 3\). Use \(\mu^3\) as the Lagrange multiplier for the household \(h = 3\) constraint \((AC)\) in Eq. 8. The first order conditions of the maximization problem in Eq. 8 are then given by:

\[
\begin{align*}
\frac{2}{p(0)e^1(0) + 2\gamma_0} - \frac{\mu^3}{p(0)e^3(0) - \gamma_0^*} &= 0, \\
\frac{2}{p(1)e^1(1) + 2\gamma_1} - \frac{\mu^3}{p(1)e^3(1) - \gamma_1^*} &= 0.
\end{align*}
\]

Solving the system of equations, the **financial equilibrium** with the pooling contract can be
calculated and is given in Table IV.

<table>
<thead>
<tr>
<th>$p_1(0)$</th>
<th>$p_2(0)$</th>
<th>$p_1(1)$</th>
<th>$p_2(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.332</td>
<td>0.668</td>
<td>0.672</td>
<td>0.328</td>
</tr>
</tbody>
</table>

$\gamma^* = (-2.163, 6.132)$

$x^1 = (1.912, 3.206, 7.026, 11.014)$ \quad $u^1 = 8.557$

$x^2 = (5.114, 7.610, 4.506, 1.979)$ \quad $u^2 = 2.595$

$x^3 = (9.974, 6.184, 1.468, 3.008)$ \quad $u^3 = 2.262$

Table IV: Equilibrium under pooling contract

**Separating** At the commodity prices found in Table IV, what would an optimal menu of separating contracts look like? The contract $\gamma(3) = \gamma(1, 3)$ can remain unchanged, but the contract $\gamma(2)$ needs to be adjusted so that both (i) $h = 2$ accepts $\gamma(2)$ and not $\gamma(3)$ and (ii) $h = 3$ continues to accept $\gamma(3)$. The optimal offer $\gamma(2)$ is determined as the solution to the following maximization problem:

$$\begin{align*}
\max_{\gamma(2) \in \mathbb{R}^2} & \quad u^1(x^1) \\
\text{subj. to} & \quad 1. \ u^2(x^2 | \gamma(2)) \geq u^2(x^2 | \gamma(3)) \\
& \quad 2. \ u^3(x^3 | \gamma(3)) = u^3(e^3) \geq u^3(x^3 | \gamma(2))
\end{align*}
$$

where the household consumption choices (as the utility functions are Cobb-Douglas) are given by

$$x^h | \gamma(k) = \begin{pmatrix}
\alpha^h(0) & \alpha^h(1) \\
(1 - \alpha^h(0)) & (1 - \alpha^h(1))
\end{pmatrix}
\begin{pmatrix}
p(0)e^h(0) - \gamma_0(k) \\
p(1)e^h(1) - \gamma_1(k)
\end{pmatrix}
\quad \text{for } h, k = 2, 3$$

and the "incentive compatibility conditions" (IC) specify that the correct household is accepting each contract.

It is not possible for the contract $\gamma(2)$ to be such that (IC) binds for both households. The optimal separating contracts are such that (IC) only binds for $h = 2$. Use $\mu^2$ as the Lagrange multiplier for the household $h = 2$ constraint (IC) in Eq. 9. The first order conditions of the maximization problem in Eq. 9 are then given by:

$$\begin{align*}
\frac{1}{p(0)e^1(0) + \gamma_0(2) + \gamma_0(3)} - \frac{\mu^2}{p(0)e^2(0) - \gamma_0(2)} &= 0. \\
\frac{1}{p(1)e^1(1) + \gamma_1(2) + \gamma_1(3)} - \frac{\mu^2}{p(1)e^2(1) - \gamma_1(2)} &= 0.
\end{align*}$$

Solving the system of equations that characterize an optimal solution to Eq. 9, the optimal
separating contracts and $h = 1$ consumption are:

<table>
<thead>
<tr>
<th>$p_1(0) = 0.332$</th>
<th>$p_2(0) = 0.668$</th>
<th>$p_1(1) = 0.672$</th>
<th>$p_2(1) = 0.328$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma(2) = (1.361, -14.831)$</td>
<td>$\gamma(3) = (-1.569, 2.122)$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x^1 = (3.807, 6.384, 0.121, 0.190)$</td>
<td>$u^1 = -4.971$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

If household $h = 1$ were to offer separating contracts, $u^1$ would be lower than what can be achieved through a pooling contract.

**Equilibrium with asymmetric information** Recall that the commodity markets are Walrasian, so the household $h = 1$ takes the commodity prices as fixed. If the household $h = 1$ were to offer a pooling contract, then the prices required to satisfy market clearing must be $p^* = (0.332, 0.668, 0.672, 0.328)$ as in Table IV. At these prices $p^*$, it is not optimal for the household $h = 1$ to offer separating contracts. However, if the household $h = 1$ were to offer separating contracts, then the prices required to satisfy market clearing would be different; call them $\bar{p}$. At these different prices $\bar{p}$, it may be the case that it is optimal to offer separating contracts and not a pooling contract. This remains to be seen, but certainly the presence of two equilibria, one involving a pooling contract at prices $p^*$ and the second involving separating contracts at prices $\bar{p}$, is a possibility.

What is certainly true is that Table IV characterizes an equilibrium in this model with asymmetric information and in this particular equilibrium, a pooling contract offer is used. Comparing the financial equilibrium under complete information in Table III and the financial equilibrium under asymmetric information in Table IV, the inclusion of this natural information gap reduces the utility of household $h = 1$ by 1%, which is equivalent to a 1.8% decrease in consumption. Additionally, the equilibrium under asymmetric information is no longer Pareto optimal.

6 Concluding Remarks

Within the canonical model of dynamic uncertainty, this paper has considered the implications of allowing strategic bargaining over the contracts transferring wealth across states of uncertainty. In this bargaining environment, contract limits play the equivalent role as a fixed asset structure does in the perfectly competitive GEI models. If the contract limits permit all households to be linked via a series of contracts, then households find it optimal to propose such contracts and all equilibrium allocations will be Pareto optimal.
One extension of the model has considered a more realistic bargaining process. Under this extension, one household in a pair selects the direction for the vector of nominal transfers, while the other household selects the vector length. No further bargaining can occur within this pair of households. This new bargaining process provides a well-defined equilibrium concept, but an appropriate First Basic Welfare Theorem cannot be stated. Namely, even if the parameters of the economy dictate that only one branch is formed, the equilibrium allocation is not guaranteed to be Pareto optimal. See the Addendum for further details.

This line of research will be extended to allow for a more general bargaining process. As the research stands currently, with the unrealistically simple process of simultaneous "take it or leave it" offers, impactful normative results have been shown and interesting applications have been considered. Moving forward, the trade-off between model complexity and tractability becomes the key modeling consideration, but some complexity is unavoidable if the bargaining process is to represent reality.

7 The Proofs

7.1 Proof of Theorem 3

Suppose not. Then $H^*_i \neq H \setminus (\bigcup_{k<i} H_k^*)$. The equilibrium allocation $(x^{h'})_{h' \in H^*_i}$ is such that

(i) $\sum_{h' \in H^*_i} (x^{h'}(s) - e^{h'}(s)) = 0 \ \forall (l, s)$, (ii) $\forall h' \in H^*_i$, $\lambda^{h'} = \kappa^{h'} \lambda^{h'}$ for some $\kappa^{h'} \in \mathbb{R}^{++}$, and

(iii) $u^{h'}(x^{h'}) \geq \tilde{u}^{h'} \ \forall h' \in H^*_i \setminus \{h_i^*\}$.

For any $h' \in H^*_i \setminus \{h_i^*\}$, $\tilde{u}^{h'}$ is determined from Eq. 5.

As $H^*_i \neq H \setminus (\bigcup_{k<i} H_k^*)$, $\exists H^*_{i+1} \neq \emptyset$ and the equilibrium allocation $(x^{h'})_{h' \in H^*_{i+1}}$ is such that

(i) $\sum_{h' \in H^*_{i+1}} (x^{h'}(s) - e^{h'}(s)) = 0 \ \forall (l, s)$, (ii) $\forall h' \in H^*_{i+1}$, $\lambda^{h'_{i+1}} = \kappa^{h'} \lambda^{h'}$ for some $\kappa^{h'} \in \mathbb{R}^{++}$, and

(iii) $u^{h'}(x^{h'}) \geq \tilde{u}^{h'} \ \forall h' \in H^*_{i+1} \setminus \{h_{i+1}^*\}$. For $h' \in H^*_{i+1} \setminus \{h_{i+1}^*\}$, $\tilde{u}^{h'}$ is determined from Eq. 5. The household $h_{i+1}^*$ must find it optimal to form such a branch $H^*_{i+1}$, rather than deviating and offering nontrivial contracts to the prior branch $H^*_i$. Thus, $u^{h_{i+1}^*}(x^{h_{i+1}^*})$, the utility obtained from making contract proposals in branch $H^*_{i+1}$, must be at least $\tilde{u}^{h_{i+1}^*}$, where $\tilde{u}^{h_{i+1}^*}$ is determined from Eq. 5.

In sum, we see that problem $(HP)$ of household $h_i^*$ is equivalent to the following maximization problem:

$$\max_{(x^{h'})_{h' \in H \setminus \bigcup_{k<i} H_k^*}} u^{h_i^*}(x^{h_i^*})$$

subj. to

1. $u^{h'}(x^{h'}) \geq \tilde{u}^{h'} \ \forall h' \in H \setminus (\bigcup_{k<i} H_k^* \cup \{h_i^*\})$. 

2. Constraint (CL) holds $\forall h' \in H \setminus (\bigcup_{k<i} H_k^*)$.

Walras’ Law is \[ \sum_{(l,s)} p_l(s) \cdot \left( \sum_{h' \in \mathcal{H} \setminus \{ k < i \}} (x_{l}^{h'}(s) - e_{l}^{h'}(s)) \right) = 0. \]

From the statement of the theorem, \( \mathcal{H} \setminus \{ k < i \} \) is achievable. Restrict attention to an economy with only households \( h' \in \mathcal{H} \setminus \{ k < i \} \). Given the vector of outside utility values \( (\tilde{u}^{h'})_{h' \in \mathcal{H} \setminus \{ k < i \}} \), there exists a unique allocation \( (y^{h'})_{h' \in \mathcal{H} \setminus \{ k < i \}} \) that is Pareto optimal. As \( \mathcal{H} \setminus \{ k < i \} \) is achievable, the allocation \( (y^{h'})_{h' \in \mathcal{H} \setminus \{ k < i \}} \) satisfies all households’ (CL) constraints.

Given the constraints \( u^{h'}(x^{h'}) \geq \tilde{u}^{h'} \quad \forall h' \in \mathcal{H} \setminus \{ \cup_{k < i} \mathcal{H}_k^* \cup \{ h_i^* \} \} \), the unique vector of maximizers to Eq. 10 (uniqueness is guaranteed from the strict convexity of the utility possibilities set given Assumption 3) is such that \( (u^{h_i^*} (x^{h_i^*}), (\tilde{u}^{h'})_{h' \in \mathcal{H} \setminus \{ \cup_{k < i} \mathcal{H}_k^* \cup \{ h_i^* \} \}}) \) lies on the utility possibilities frontier. Thus the maximizers of Eq. 10, \( (x^{h'})_{h' \in \mathcal{H} \setminus \{ \cup_{k < i} \mathcal{H}_k^* \}} \), are equivalent to the vector \( (y^{h'})_{h' \in \mathcal{H} \setminus \{ \cup_{k < i} \mathcal{H}_k^* \}} \), a Pareto optimal allocation in which all households’ (CL) constraints are satisfied. The offered contracts in turn are such that \( \forall h' \in \mathcal{H} \setminus \{ k < i \} \), \( \lambda^{h'} = \kappa^{h'} \lambda^{h'} \) for some \( \kappa^{h'} \in \mathbb{R}_{++} \). This implies \( \mathcal{H}_i^* = \mathcal{H} \setminus \{ k < i \} \), finishing the argument.

### 7.2 Proof of Theorem 4

The proof proceeds in two steps to construct a financial equilibrium. Step 1, using the fact that any financial equilibrium allocation is Pareto optimal, finds the real variables: household consumption \( (x^{h})_{h \in \mathcal{H}} \) and commodity prices \( p \). Step 2 finds the financial variables: contract proposals \( (\tilde{\gamma}^{h})_{h \in \mathcal{H}} \).

**Step 1:** Given Corollary 1, the allocation and commodity prices of a financial equilibrium are constructed as the fixed point of an appropriate function. Define the price set as \( p = (p(s))_{s \in \mathcal{S}} \in (\Delta^{L-1})^{S+1} \), a compact and convex set. For all \( h > 1 \), define the set of outside utility values such that \( \tilde{u}^h \in \tilde{U}^h = [u^h(\sigma^h(0)), u^h(\sum_{h \in \mathcal{H}} e^h)] \). Then \( (\tilde{u}^h)_{h>1} \in \tilde{U} = \times_{h>1} \tilde{U}^h \), a compact and convex set. I will define the self map \( \Psi : (\Delta^{L-1})^{S+1} \times \tilde{U} \rightarrow (\Delta^{L-1})^{S+1} \times \tilde{U} \).

The mapping \( \Psi \) is composed of two functions.

1. The first function \( \Psi_p : \tilde{U} \rightarrow (\Delta^{L-1})^{S+1} \) works in two stages: (i) given \( (\tilde{u}^h)_{h>1} \), the unique Pareto optimal allocation \( (x^{h})_{h \in \mathcal{H}} \) of the economy is determined, and (ii) given the Pareto optimal allocation \( (x^{h})_{h \in \mathcal{H}} \), the commodity prices \( p = (p(s))_{s \in \mathcal{S}} \) supporting the allocation \( (x^{h})_{h \in \mathcal{H}} \) can be found. Stage (ii) is defined by the continuous equations (think Second Basic Welfare Theorem):

\[
p_l(s) = \frac{D_{(l,s)}u^1(x^1)}{\sum_{l'} D_{(l',s)}u^1(x^1)} \quad \forall (l, s).
\]
Stage (i) is defined as the implicit function of the following system of equations:

\[
F \left( \left( x^h \right)_{h \in H}, \left( k^h \right)_{h > 1}, \left( \tilde{u}^h \right)_{h > 1} \right) = \left\{ \begin{array}{ll}
\forall h > 1: & Du^1 \left( x^1 \right) - k^h Du^h \left( x^h \right) \\
\sum_{h \in H} e^h - \sum_{h \in H} x^h \\
\forall h > 1: & u^h \left( x^h \right) - \tilde{u}^h
\end{array} \right. = 0.
\]

I will show that \( D_{x,\kappa}F \left( \cdot \right) \) is invertible. To do this, I set the equation

\[
\left( \left( \Delta x^T_h \right)_{h > 1}, \Delta p^T, \left( \Delta k^h \right)_{h > 1} \right) D_{x,\kappa}F \left( \cdot \right) = 0,
\]

and must verify that \( \left( \left( \Delta x^T_h \right)_{h > 1}, \Delta p^T, \left( \Delta k^h \right)_{h > 1} \right) = 0 \). Eq. 11 is given by:

\[
\begin{align*}
\left( \sum_{h>1} \Delta x^T_h \right) D^2 u^1 \left( x^1 \right) - \Delta p^T &= 0 \quad (12.a) \\
-\kappa^h \cdot \Delta x^T_h D^2 u^h \left( x^h \right) + \Delta k_h Du^h \left( x^h \right) - \Delta p^T &= 0 \quad \forall h > 1 \quad (12.b) \\
-\Delta x^T_h \left[ Du^h \left( x^h \right) \right]^T &= 0 \quad \forall h > 1 \quad (12.c)
\end{align*}
\]

Post-multiply Eq. 12.b by \( \Delta x_h \) and use Eq. 12.c to obtain:

\[
\kappa^h \cdot \Delta x^T_h D^2 u^h \left( x^h \right) \Delta x_h + \Delta p^T \Delta x_h = 0 \quad \forall h > 1 .
\]

Summing Eq. 13 over all households \( h > 1 \), and using Eq. 12.a to replace the term \( \Delta p^T \), we obtain:

\[
\sum_{h>1} \kappa^h \cdot \Delta x^T_h D^2 u^h \left( x^h \right) \Delta x_h + \left( \sum_{h>1} \Delta x^T_h \right) D^2 u^1 \left( x^1 \right) \left( \sum_{h>1} \Delta x_h \right) = 0.
\]

From Eq. 14 and Assumption 3 \( \left( Du^h \left( x^h \right) \right) >> 0 \) and \( D^2 u^h \left( x^h \right) \) negative definite), \( \kappa^h > 0 \) \( \forall h > 1 \) and \( \left( \Delta x^T_h \right)_{h>1} = 0 \). From Eqs. 12.a and 12.b, then \( \left( \Delta p^T, \left( \Delta k^h \right)_{h>1} \right) = 0 \). Thus, \( D_{x,\kappa}F \left( \cdot \right) \) is invertible, so we apply the Implicit Function Theorem. The implicit function \( \left( x^h \right)_{h \in H} = f \left( \left( \tilde{u}^h \right)_{h > 1} \right) \) is \( C^1 \). Therefore, the function \( \Psi_p \) mapping \( \left( \tilde{u}^h \right)_{h > 1} \mapsto p \) (by way of the allocation \( \left( x^h \right)_{h \in H} \)) is \( C^1 \).

2. The second function \( \Psi_u : (\Delta L^{-1})^{S+1} \rightarrow \tilde{U} \) works by backward induction in \( (H - 1) \) stages: (i) \( \tilde{u}^H = u^H (s^H(0)) \) and (ii) for any \( h : 1 < h < H \), given the commodity prices \( p \) and \( \left( \tilde{u}^{h'} \right)_{h' > h} \), the value \( \tilde{u}^h \) is the utility that household \( h \) can receive by only making contract proposals to households \( h' > h \), subject to its contract limit.
\[ \eta^h \] Specifically, \( \tilde{u}^h = \max_{\mathcal{H}^\# \subseteq \{ h+1, \ldots, H \}; \# \mathcal{H}^\# \leq h^b} u^h (\hat{x}_{\mathcal{H}^\#}^h) \), where the maximum is over all subsets of households \( \mathcal{H}^\# \subseteq \{ h+1, \ldots, H \} \) that household \( h \) can offer nontrivial contracts to (subject to the contract limit \( \eta^h \)). For any one subset of households \( \mathcal{H}^\# \), the utility \( u^h (\hat{x}_{\mathcal{H}^\#}^h) \) is determined as the implicit function of the following system of equations, with variables \( (\hat{x}, \lambda, \kappa) = \left( (\hat{x}_{h'}^h)_{h' \in \mathcal{H}^{\#} \cup \{ h \}}, (\lambda_{h'}^h)_{h' \in \mathcal{H}^{\#} \cup \{ h \}}, (\kappa_{h'}^h)_{h' \in \mathcal{H}^{\#}} \right) \) and parameters \( \tilde{u} = (\tilde{u}_{h'})_{h' \in \mathcal{H}^{\#}} \), where I have omitted the subscript \( \mathcal{H}^\# \) for simplicity:

\[
\Phi_h^* (\hat{x}, \lambda, \kappa; p, \tilde{u}) = \begin{cases} 
\forall h' \in \mathcal{H}^{\#} \cup \{ h \} : & \left[ Du^{h'} (\hat{x}^{h'}) - \lambda^{h'} P \right]^T \\
p(s) \sum_{h' \in \mathcal{H}^{\#} \cup \{ h \}} (\hat{x}^{h'} (s) - e^{h'} (s)) = 0 \\
\forall h' \in \mathcal{H}^{\#} : & \left[ \lambda^h - \kappa^{h'} \lambda^{h'} \right]^T \\
\forall h' \in \mathcal{H}^{\#} : & u^{h'} (\hat{x}^{h'}) - \tilde{u}^{h'} 
\end{cases} = 0.
\]

Notice that market clearing conditions are not included in \( \Phi_h^* \), because the allocation \( (\hat{x}_{h'}^h)_{h' \in \mathcal{H}^{\#} \cup \{ h \}} \) need not be an equilibrium allocation, but only an allocation obtained when \( h \) deviates and rejects all contract proposals. I will show that \( D_{\hat{x}, \lambda, \kappa} \Phi_h^* (\cdot) \) is invertible. To do this, I set the equation

\[
\left( (\Delta x_{h'}^T)_{h' \in \mathcal{H}^{\#} \cup \{ h \}}, \Delta \lambda^T, (\Delta \kappa_{h'}^T)_{h' \in \mathcal{H}^{\#}}, (\Delta u^{h'})_{h' \in \mathcal{H}^{\#}} \right) D_{\hat{x}, \lambda, \kappa} \Phi_h^* (\cdot) = 0, \tag{15}
\]

and must verify that \( \left( (\Delta x_{h'}^T)_{h' \in \mathcal{H}^{\#} \cup \{ h \}}, \Delta \lambda^T, (\Delta \kappa_{h'}^T)_{h' \in \mathcal{H}^{\#}}, (\Delta u^{h'})_{h' \in \mathcal{H}^{\#}} \right) = 0. \) Eq. 15 is given by:

\[
\begin{align*}
\Delta x_h^T D^2 u^h (x^h) + \Delta \lambda^T P &= 0 \tag{16.a} \\
\Delta x_{h'}^T D^2 u^{h'} (x^{h'}) + \Delta \lambda^T P + \Delta u^{h'} D u^{h'} (x^{h'}) &= 0 \quad \forall h' \in \mathcal{H}^{\#} \tag{16.b} \\
- \Delta x_{h'}^T P + \sum_{h' \in \mathcal{H}^{\#}} \Delta \kappa_{h'}^T &= 0 \tag{16.c} \\
- \Delta x_{h'}^T P - \kappa_{h'}^T \Delta \kappa_{h'}^T &= 0 \quad \forall h' \in \mathcal{H}^{\#} \tag{16.d} \\
\Delta \kappa_{h'}^T \left[ \lambda^h \right]^T &= 0 \quad \forall h' \in \mathcal{H}^{\#} \tag{16.e}
\end{align*}
\]

\[12\text{In truth, the value } \tilde{u}^h \text{ is determined as the maximum of all outside utility values that } h \text{ can receive by only making contract proposals to households } h' > h, \text{ where the maximum is over the set of all permutations } \pi : \{ h+1, \ldots, H \} \rightarrow \{ h+1, \ldots, H \}. \text{ The mapping } \pi \text{ permutes the order of households } h' > h, \text{ without changing the endowments and utility functions of these households. The reason that we consider such permutations is that } \left( \tilde{u}^{h'} \right)_{h' > h} \text{ are variables for household } h \text{ as in the household problem } (HP2) \text{ in Subsection 4.1.1.} \]
Post-multiply Eq. 16.b by $\Delta x_{h'}$ and use Eqs. 16.d and 16.e, together with $Du' (x') = \lambda' P$ from $\Phi_h^* (\cdot) = 0$, to arrive at:

$$\frac{1}{\kappa'} \Delta x_{h'}^T D^2 u' (x') \Delta x_{h'} = \Delta \lambda^T \Delta \kappa_{h'}^T \forall h' \in H^\#.$$  \hspace{1cm} (17)

From $\Phi_h^* (\cdot) = 0$, $\kappa' > 0$. Now, post-multiply Eq. 16.a by $\Delta x_h$ and use Eq. 16.c to obtain:

$$\Delta x_h^T D^2 u (x^h) \Delta x_h + \Delta \lambda^T (\sum_{h' > h} \Delta \kappa_{h'}^T) = 0.$$  \hspace{1cm} (18)

Summing Eq. 17 over all $h' \in H^\#$ and combining with Eq. 18 yields:

$$\Delta x_h^T D^2 u (x^h) \Delta x_h + \sum_{h' \in H^\#} \left( \frac{1}{\kappa'} \Delta x_{h'}^T D^2 u' (x') \Delta x_{h'} \right) = 0.$$  \hspace{1cm} (19)

From Eq. 19 and Assumption 3 ($D^2 u (x^h)$ is negative definite), then $\left( \Delta x_{h'}^T \right)_{h' \in H^\# \cup \{ h \}} = 0$. From Eqs. 16.d, $\left( \Delta \kappa_{h'}^T \right)_{h' \in H^\#} = 0$. From Eqs. 16.a and 16.b, $\left( \Delta \lambda^T, (\Delta u_{h'})_{h' \in H^\#} \right) = 0$. Thus, $D_{x, \lambda, \kappa} \Phi_h^* (\cdot)$ is invertible, so we apply the Implicit Function Theorem. The implicit function $x_{h^\#}^h = \phi_h^* \left( p, (\tilde{u}_{h'})_{h' \in H^\#} \right)$ is $C^1$. Thus, $\tilde{u}_h = \max_{H^\# \subseteq \{ h+1, \ldots, H \} : \# H^\# \leq h} u_h (x^h)$ is a $C^1$ function of $\left( (\tilde{u}_{h'})_{h' \in H^\#} \right)$. Proceeding by backward induction, then $\tilde{u} = (\tilde{u}_{h'})_{h' > h}$ is a $C^1$ function of $p$. Therefore, the function $\Psi$ mapping $p \mapsto (\tilde{u}_h)_{h > h}$ is $C^1$.

Define the mapping $\Psi = (\Psi_p, \Psi_u) : (\Delta L^{-1})^{S+1} \times \tilde{U} \rightarrow (\Delta L^{-1})^{S+1} \times \tilde{U}$. By definition, if $\left( p, (\tilde{u}_h)_{h > h} \right) = \Psi \left( p, (\tilde{u}_h)_{h > h} \right)$, then for the corresponding allocation $(x^h)_{h \in H} = f \left( (\tilde{u}_{h'})_{h > h} \right)$, we have located a financial equilibrium allocation $(x^h)_{h \in H}$ and the corresponding vector of commodity prices $p$. The mapping $\Psi$ is a continuous self-map over a convex and compact set, so Brouwer’s Fixed Point Theorem guarantees the existence of a fixed point.

**Step 2:** There may be multiple contract orderings that can support the same financial equilibrium allocation $(x^h)_{h \in H}$ and commodity prices $p$ (an example of this is shown in Figures 1(a) and 1(b)). Select one ordering such that $(CL)$ is satisfied $\forall h \in H$. Then, starting at the end of the branch $H = H'_1$, proceed by backward induction to define the nontrivial contract proposals $(\tilde{\gamma}'(h, h') : \tilde{\gamma}'(h, h') \neq 0)_{h, h' \in H}$ so that the budget constraints are satisfied:

$$P(e^h - x^h) + \sum_{h' > h} \tilde{\gamma}'(h, h') - \sum_{h' < h} \tilde{\gamma}'(h', h) = 0 \forall h \in H.$$

The remaining contract proposals in $(\tilde{\gamma}^h)_{h \in H}$ can be defined as trivial contracts. The vector $(x^h, \tilde{\gamma}^h)_{h \in H}, p)$ is a financial equilibrium.
7.3 Proof of Theorem 5

The definition of a branch remains valid for unconstrained financial equilibria. Suppose, for contradiction, that an unconstrained financial equilibrium exists in which household \( h \) makes a nontrivial contract proposal \( \tilde{\gamma} (h, h') \) that is rejected by household \( h' \). We know that \( h \) is indifferent between making this proposal and making the trivial contract proposal \( \tilde{\gamma} (h, h') = 0 \). What we have to verify is that the choice to propose nontrivial \( \tilde{\gamma} (h, h') \) cannot support additional equilibria.

The main result is that upon viewing the nontrivial contract proposal \( \tilde{\gamma} (h, h') \), all households know that the proposal will be rejected, so its presence has no effect on any of the other proposed contracts. There are two cases to consider.

**Case I**: \( h, h' \in \mathcal{H}_i^* \) for some \( i \), even if the contract \( \tilde{\gamma} (h, h') \) is rejected.

From the analysis in Subsection 3.1, \( h, h' \in \mathcal{H}_i^* \) without a direct contract connection implies \( \lambda^h = \kappa^{h'} \lambda^{h'} \) for some \( \kappa^{h'} \in \mathbb{R}_{++} \). If the contract proposal is accepted by household \( h' \), then this must necessarily make \( h \) strictly worse off. And if the contract were to make \( h \) strictly better off, then it would not be accepted by \( h' \) (as it would make \( h' \) strictly worse off). Then all households ignore the contract proposal as they know it will not be accepted.

**Case II**: If the contract is rejected, then \( \tilde{\gamma} (h, h') \) is rejected.

If the acceptance of this contract would cause either \( h \) or \( h' \) to violate its contract limit (CL) (with the mandated infinitely large utility loss for such an invalid choice), then all households ignore the contract proposal as they know it will not be accepted. Yet, if the contract can be accepted and both households \( h \) and \( h' \) continue to satisfy (CL), then a larger branch \( i^+ \) such that \( h, h' \in \mathcal{H}_i^* \) is achievable. Citing Theorem 3, both households will belong to the same branch in equilibrium, which is a contradiction of Case II.

7.4 Proof of Theorem 6

I will show that generically on \( (e^h)_{h \in \mathcal{H}}, \) the matrix \( D_{\xi} \Phi (\xi, \theta)_{|\Phi (\xi, \theta) = 0} \) has full row rank. The matrix \( D_{\xi} \Phi (\xi, \theta)_{|\Phi (\xi, \theta) = 0} \) is given below (where the rows correspond to equations of \( \Phi \) and the columns correspond to variables \( \xi = (x^h, \lambda^h)_{h \in \mathcal{H}}, (\tilde{\gamma}^h)_{h=h_1,\ldots,h_I}, (\mu^h)_{h \in \mathcal{H}\setminus\{h_1,\ldots,h_I\}} \). To conserve on space, I employ the following conventions:

\[
\begin{align*}
c \left( A^h \right) &= \begin{pmatrix} A^1 \\ \vdots \\ A^H \end{pmatrix}, & r \left( A^h \right) &= \begin{pmatrix} A^1 & \cdots & A^H \end{pmatrix}, & d \left( A^h \right) &= \begin{pmatrix} A^1 & 0 & 0 \\ 0 & \cdots & 0 \\ 0 & 0 & A^H \end{pmatrix} \end{align*}
\]
where \((c, r, d)\) stand for column, row, and diagonal, respectively. The matrix \(D_\xi \Phi (\xi, \theta)_{|\Phi(\xi, \theta)=0}\) is given by:

\[
\begin{pmatrix}
  d (D^2 u^h) & d(-P^T) & 0 & 0 & c (\Lambda_2^h) \\
  d(-P) & 0 & \Lambda_3 & 0 & c (Z^h) \\
  0 & \Lambda_3^* & 0 & \Lambda_4 & 0 \\
  0 & 0 & \Lambda_4^* & 0 & 0 \\
  r (-\Lambda) & 0 & 0 & 0 & 0
\end{pmatrix}
\]

where \(A\) is the \((G - S - 1) \times G\) matrix \(A = \begin{pmatrix} I_{L-1} & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & (I_{L-1} & 0) \end{pmatrix}\), \(\Lambda_2^h\) is the \(G \times (G - S - 1)\) matrix \(\Lambda_2 = \begin{pmatrix} \lambda^h(0)I_{L-1} & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & \lambda^h(S)I_{L-1} \end{pmatrix}\), and \(Z^h\) is the \((S + 1) \times (G - S - 1)\) matrix \(Z^h = \begin{pmatrix} (e_{\lambda, L}^h(0) - x_{\lambda, L}^h(0))^T & 0 & 0 \\ 0 & \ldots & 0 \\ 0 & 0 & (e_{\lambda, L}^h(S) - x_{\lambda, L}^h(S))^T \end{pmatrix}\). For the derivatives of \((BC')\) with respect to \(\hat{\gamma}_h\) \(h = h_1^*, \ldots, h_4^*\), \(\Lambda_3 = \begin{pmatrix} 1 & \ldots & 1 \\ 0 & \ldots & 0 \\ 0 & 0 & -1_{S+1} \end{pmatrix}\). For the derivatives of \((FOC\hat{\gamma})\) with respect to \(\lambda^h\) \(h \in H\), \(\Lambda_3^* = \begin{pmatrix} 1 & \ldots & 1 \\ 0 & \ldots & 0 \\ 0 & 0 & -1_{S+1} \end{pmatrix}\). For the derivatives of \((FOC\hat{\gamma})\) with respect to \(\mu^h\) \(h \in H \setminus \{h_1, \ldots, h_4\}\), \(\Lambda_4 = \begin{pmatrix} \ldots & 0 & 0 \\ 0 & -\lambda^h & \ldots \\ 0 & 0 & \ldots \end{pmatrix}\).
where \( h \in \mathcal{H}_1 \setminus \{ h_1^* \} \). From the analysis in Subsection 3.1, for the derivatives of \((AC)\) with respect to \((\gamma_h)_{h=h_1^*,...,h_I^*}\), \( \Lambda^*_A = \begin{bmatrix} ... & 0 & 0 \\ 0 & -\lambda^h & 0 \\ 0 & 0 & ... \end{bmatrix} \) for \( h \in \mathcal{H}_1 \setminus \{ h_1^* \} \).

To show that generically on \((e_h)_{h \in \mathcal{H}}\), the matrix \( D_{\xi} \Phi (\xi, \theta) |_{\Phi (\xi, \theta) = 0} \) has full row rank, I have to show that the extended matrix

\[
M = \left( D_{\xi} \Phi (\xi, \theta) |_{\Phi (\xi, \theta) = 0} \mid D_e \Phi (\xi, \theta) |_{\Phi (\xi, \theta) = 0} \right)
\]

has full row rank. To show that \( M \) has full row rank, premultiply by the row vector

\[
\nu^T = (\Delta x^T, \Delta \lambda^T, \Delta \hat{\gamma}^T, \Delta \mu^T, \Delta p^T) \in \mathbb{R}^n.
\]

The theorem is proved upon showing that \( \nu^T = 0 \). For convenience, the vector \( \nu^T \) is divided into the indicated subvectors which correspond sensibly with the following equations of \( \Phi \):

\[
\begin{align*}
\Delta x^T & \iff FOC x \\
\Delta \lambda^T & \iff BC \\
\Delta \hat{\gamma}^T & \iff FOC \hat{\gamma} \\
\Delta \mu^T & \iff AC \\
\Delta p^T & \iff MC x.
\end{align*}
\]

I will list the equations of \( \nu^T M = 0 \) in the order that is most convenient to obtain \( \nu^T = 0 \). At my disposal is the system of equations \( \Phi (\xi, \theta) = 0 \).

**First**, for the columns corresponding to derivatives with respect to \((x^h)_{h \in \mathcal{H}}\) and \((e^h)_{h \in \mathcal{H}}\):

\[
(\Delta x^h)^T D^2 u^h (x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda = 0
\]

\[
(\Delta \lambda^h)^T P + \Delta p^T \Lambda = 0.
\]

From the definition of \( \Lambda \), \( (\Delta \lambda^h)^T = 0 \) \( \forall h \in \mathcal{H} \) and \( \Delta p^T = 0 \). Further, \((\Delta x^h)^T D^2 u^h (x^h) \Delta x^h = 0 \) \( \forall h \in \mathcal{H} \). From Assumption 3, \((\Delta x^h)^T = 0 \) \( \forall h \in \mathcal{H} \).

**Second**, for the columns corresponding to derivatives with respect to \((\lambda^h)_{h \notin \{ h_1^*,...,h_I^* \}}\), using the definition of \( \Lambda^*_\lambda \), \( (\Delta \hat{\gamma}^h)^T = 0 \) \( \forall h = h_1^*,...,h_I^* \). For the columns corresponding
to derivatives with respect to \((\hat{z}^h)_{h=h_1^*,...,h_I^*}\), using the definition of \(\Lambda^*_k\), \(\Delta \mu^h = 0 \forall h \notin \{h_1^*,...,h_I^*\}\).

In conclusion, \(\nu^T = 0\) and the proof of Theorem 6 is complete.

### 7.5 Proof of Corollary 2

I will show that generically on \((e^h)_{h \in \mathcal{H}'}\), the matrix \(D_{\xi} \left( \begin{pmatrix} \Phi(\xi, \theta) |_{\Phi(\xi, \theta) = 0} \\ \rho(\xi, \theta) \end{pmatrix} \right)\) has full row rank where \(\rho(\xi, \theta) : \Xi \times \Theta \rightarrow \mathbb{R}\) will either be an equation \(\rho(\xi, \theta) = \sum_{h \in \mathcal{H}'} (e^h_i(s) - x_1^h(s))\) for some \((\mathcal{H}', s)\) [for fact (i)] or an equation \(\rho(\xi, \theta) = \lambda^h(1)\lambda^{h'}(0) - \lambda^h(0)\lambda^{h'}(1)\) for some \((h, h')\) [for fact (ii)]. The derivative matrix \(D_{\xi} \left( \begin{pmatrix} \Phi(\xi, \theta) |_{\Phi(\xi, \theta) = 0} \\ \rho(\xi, \theta) \end{pmatrix} \right)\) has more rows than columns, so showing that the matrix \(D_{\xi} \left( \begin{pmatrix} \Phi(\xi, \theta) |_{\Phi(\xi, \theta) = 0} \\ \rho(\xi, \theta) \end{pmatrix} \right)\) has full row rank over a generic subset of \(\mathcal{E}\) is sufficient to prove that the extra equation \(\rho(\xi, \theta) = 0\) can never hold for \(\theta \in \Theta\), where \((e^h)_{h \in \mathcal{H}}\) lies in a generic subset of \(\mathcal{E}\).

For fact (i), I set \((\nu^T, r) \cdot D_{\xi} \left( \begin{pmatrix} \Phi(\xi, \theta) |_{\Phi(\xi, \theta) = 0} \\ \rho(\xi, \theta) \end{pmatrix} \right) = 0\) where the vector \(\nu^T\) is as defined in the proof of Theorem 6 (Subsection 7.4). As \(\mathcal{H} \neq \mathcal{H}'\), then \(\exists h^* \notin \mathcal{H}'\). From the columns corresponding to derivatives with respect to \(x^{h^*}\) and \(e^{h^*}\), as in the proof of Theorem 6, \((\Delta x^{h^*})^T = (\Delta \lambda^{h^*})^T = \Delta p^T = 0\). For the columns corresponding to the derivatives with respect to \((x^h)_{h \neq h^*}\) and \((e^h)_{h \neq h^*}\):

\[
(\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta r G = 0 \\
(\Delta \lambda^h)^T P + \Delta r G = 0,
\]

where \(G_1(s) = 1\) and \(G_i(s') = 0 \forall (l, s') \neq (1, s)\). Thus, \((\Delta \lambda^h)^T = 0 \forall h \neq h^*\) and \(\Delta r = 0\).

The remaining steps in the proof of Theorem 6 remain valid.

For fact (ii), \(I > 1\) is assumed. Define \(h = h_1^*\) and \(h' = h_2^*\). Notice in the proof of Theorem 6 that the columns corresponding to derivatives with respect to \(\left(\lambda^h, \lambda^{h'}\right)\) are not employed. Thus, as in the proof of Theorem 6, \(\nu^T = 0\) and the columns for the first order conditions with respect to \(\left(\lambda^h, \lambda^{h'}\right)\) can be used to show that the additional equation \(\rho(\xi, \theta) = \lambda^h(1)\lambda^{h'}(0) - \lambda^h(0)\lambda^{h'}(1)\) is independent from the other equations \(\Phi(\xi, \theta)\). This completes the argument.
7.6 Proof of Theorem 7

The proof of this theorem will follow the framework of Citanna, Kajii, and Villanacci (1998), henceforth simply CKV.

Picking a vector of parameters \( \tilde{\theta} = (e^h, \bar{u}^h)_{h \in \mathcal{H}} \) such that \( (e^h)_{h \in \mathcal{H}} \) belongs to a generic subset of \( \mathcal{E} \), then all resulting parameters \( \tilde{\theta} \) are regular values of \( \pi \). In particular, this means that there exists an open set \( \Theta' \) around \( \tilde{\theta} \) such that any parameters \( \theta \in \Theta' \) are also regular values of \( \pi \). Denote \( \tilde{\xi} \) such that \( \pi^{-1}(\tilde{\theta}) = (\tilde{\xi}, \tilde{\theta}) \). The set of allocations \( (x^h)_{h \in \mathcal{H}} \) in a local neighborhood around \( (\bar{x}^h)_{h \in \mathcal{H}} \) such that \( U(x) = (u^1(x_1), ..., u^H(x^H)) \) \( \gg \) \( U(\bar{x}) = (u^1(\bar{x}_1), ..., u^H(\bar{x}_H)) \) is open. If some \( I \)-feasible transfer \( (\tau^h)_{h \in \mathcal{H}} \) results in an allocation that Pareto dominates \( (\bar{x}^h)_{h \in \mathcal{H}} \), then all \( I \)-feasible transfers in an open neighborhood around \( (\tau^h)_{h \in \mathcal{H}} \) result in an allocation that Pareto dominates \( (\bar{x}^h)_{h \in \mathcal{H}} \).

Take as given a financial equilibrium \( ((x^h, \lambda^h)_{h \in \mathcal{H}}, (\tilde{\xi}^h)_{h = h_1^*, ..., h_I^*}, (\mu^h)_{h \in \mathcal{H}\setminus\{h_1^*, ..., h_I^*\}}, P) \). Given parameters \( \theta = (e^h, u^h)_{h \in \mathcal{H}} \), the variables \( \dot{\xi} = ((\dot{x}^h, \dot{\lambda}^h)_{h \in \mathcal{H}}, \dot{\mu}^h) \) and \( I \)-feasible transfers \( \tau = (\tau^h)_{h \in \mathcal{H}} \) are such that \( \dot{x} = (\dot{x}^h)_{h \in \mathcal{H}} \) is a constrained feasible allocation iff \( \Gamma(\dot{\xi}, \tau, \theta) = 0 \), where \( \Gamma : \times_{h \in \mathcal{H}} (X^h \times \mathbb{R}^{\mathcal{S}_+ + 1}) \times \mathbb{R}^{G-(S+1)} \times \mathbb{R}^{H(S+1)} \times \Theta \rightarrow \mathbb{R}^m \) is defined by the \( m = H(G + S + 1) + G - (S + 1) + I(S + 1) \) equations:

\[
\begin{pmatrix}
FOC_x \\
BC \\
MC_x \\
FC_T
\end{pmatrix}
= 
\begin{pmatrix}
\left( D_{x^h}(\dot{x}^h) - \dot{\lambda}^h \dot{P} \right)_{h \in \mathcal{H}} \\
\dot{P} (e^h - \dot{x}^h) + \tau^h_{h \in \mathcal{H}} \\
\left( \sum_{h \in \mathcal{H}} (e^h_{\lambda_i}(s) - \dot{x}^h_{\lambda_i}(s)) \right)_{s \in S} \\
\left( \sum_{h \in \mathcal{H}^*} \tau^h \right)_{i = 1, ..., I}
\end{pmatrix}.
\]

The analysis is conducted evaluating functions where \( \forall i = 1, ..., I \), \( \tau^h = -\dot{\gamma} (h_i^*, h) \) \( \forall h \in \mathcal{H}^*_{i} \setminus \{h_i^*\} \) and \( \tau^+h^* = \sum_{h \in \mathcal{H}^*_{i} \setminus \{h_i^*\}} \dot{\gamma} (h_i^*, h) \). By definition, if \( \Gamma(\dot{\xi}, \tau^*, \theta) = 0 \) and \( \Phi(\dot{\xi}, \theta) = 0 \),

then \( \dot{\xi} = (x^h, \lambda^h)_{h \in \mathcal{H}}, P) \).

Define the \( (H + m) \times (H(G + S + 1) + G - (S + 1) + H(S + 1)) \) matrix \( \Psi_0 : \)

\[
\Psi_0(\dot{\xi}, \tau, \theta) = 
\begin{pmatrix}
D_\xi U(\dot{x}) \\
0
\end{pmatrix}
\begin{pmatrix}
D_\xi \Gamma(\dot{\xi}, \tau, \theta)
\end{pmatrix}

= \begin{pmatrix}
D_\xi \Gamma(\dot{\xi}, \tau, \theta)
\end{pmatrix}
\begin{pmatrix}
\Psi_0(\dot{\xi}, \tau, \theta)
\end{pmatrix}.
\]

From CKV, if \( \Psi_0 \) has full row rank, \( \exists \dot{\xi} \neq (x^h, \lambda^h)_{h \in \mathcal{H}}, P \) s.t. \( \dot{\xi} \) satisfies \( \Gamma(\dot{\xi}, \tau, \theta) = 0 \) (for some \( \tau \)) and \( U(\dot{x}) > U(x) \). To have full row rank, there must exist fewer rows than columns,
so \( H + I(S + 1) \leq H(S + 1) \) or

\[
H \leq (H - I)(S + 1). 
\]  
(20)

As the number of branches is bounded, \( I \leq \frac{H}{2} \) (as discussed in Subsection 3.2), then Eq. 20 is trivially satisfied:

\[
(H - I)(S + 1) \geq \frac{H}{2} (S + 1) \geq H. 
\]

If the matrix \( \Psi_0 \) has more columns than rows, I remove some columns (it does not matter which) in order to obtain a square matrix \( \Psi \). This matrix \( \Psi \) does not have full rank iff \( \exists \nu \in \mathbb{R}^{H+m} \) s.t. \( \Phi'(\xi, \tau^*, \nu, \theta) = 0 \) where

\[
\Phi'(\xi, \tau^*, \nu, \theta) = \begin{pmatrix} \Psi^T \nu \\ \nu^T \nu/2 - 1 \end{pmatrix}.
\]

For simplicity, I divide the vector \( \nu^T \) into subvectors that each represent a certain equation in \( \Psi \). Define \( \nu^T = (\Delta u^T, \Delta x^T, \Delta \lambda^T, \Delta p^T, \Delta \tau^T) \in \mathbb{R}^{H+m} \), where each subvector corresponds sensibly to an equation (row) in \( \Psi \) as follows:

- \( \Delta u^T \leftrightarrow D_\xi U(\dot{x}) \)
- \( \Delta x^T \leftrightarrow FOCx \)
- \( \Delta \lambda^T \leftrightarrow BC \)
- \( \Delta p^T \leftrightarrow MCx \)
- \( \Delta \tau^T \leftrightarrow FC\tau. \)

I will have proven Theorem 7 if I can show that for a generic choice of \( \theta \in \Theta \), there does not exist \( (\xi, \nu) \) s.t.

\[
\Phi(\xi, \theta) = 0 \quad (21)
\]

\[
\Phi' \left( \left( (x^h, \lambda^h)_{h \in \mathcal{H}}, p \right), \tau^*, \nu, \theta \right) = 0.
\]

Counting equations and unknowns, Eq. 21 has \( n \) equations in \( \Phi \), \( n \) variables \( \xi \), \( H + m + 1 \) equations in \( \Phi' \), and only \( H + m \) variables \( \nu \). I must show that over a generic subset of parameters \( \Theta \), the derivative matrix \( D_{\xi,\nu} \begin{pmatrix} \Phi \\ \Phi' \end{pmatrix} \) has full row rank. I refer to the \((ND)\) condition (given in Eq. 22 below) of CKV, which is a sufficient condition for the full row rank of \( D_{\xi,\nu} \begin{pmatrix} \Phi \\ \Phi' \end{pmatrix} \). The condition states that for \( \tau = \tau^* \) and \( \dot{\xi} = \left( (x^h, \lambda^h)_{h \in \mathcal{H}}, p \right) \), the
matrix
\[
\begin{pmatrix}
P^T \\
\nu^T \\
\end{pmatrix} D_A \Phi'
\]
has full row rank. \( (22) \)

For simplicity, I break up the analysis into two cases: Case I: \((\Delta x^h)^T \neq 0 \ \forall h \in \mathcal{H}\) and Case II: \((\Delta x^h)^T = 0\) for some \(h \in \mathcal{H}\). In Case I, I show that Eq. 22 holds over a generic subset of \(\Theta\). In Case II, I show that, for all \((u^h)_{h \in \mathcal{H}} \in \mathcal{U}\) and all \((e^h)_{h \in \mathcal{H}}\) in a generic subset of \(\mathcal{E}\), Eq. 21 does not have a solution.

7.6.1 Case I: \((\Delta x^h)^T \neq 0 \ \forall h \in \mathcal{H}\)

Lemma 1 For \(\tau = \tau^*\), then \(D_u \Phi' = \begin{pmatrix} d(\hat{A}^h) \end{pmatrix}\) where \(d(\hat{A}^h)\) has full row rank and corresponds to the rows for derivatives with respect to \((x^h)_{h \in \mathcal{H}}\).

Proof. See Subsection 7.6.3. ■

The matrix \(\begin{pmatrix} P^T & D_A \Phi' \end{pmatrix}\) is given below (where the rows correspond to the variables \(\{(x^h, \lambda^h)_{h \in \mathcal{H}}, p\}\), \(I\)-feasible transfers \((\tau^h)_{h \in \mathcal{H}}\), and vector \(v^T\) in that order). I employ the same \((c, r, d)\) convention as in the proof of Theorem 6. The matrix \(\begin{pmatrix} P^T & D_A \Phi' \end{pmatrix}\) is given by:

\[
\begin{pmatrix}
d(\hat{D} u^h(x^h)^T) & d(D^2 u^h) & d(-P^T) & c(-\Lambda^T) & 0 & d(\hat{A}^h) \\
0 & d(-P) & 0 & 0 & 0 & 0 \\
0 & r(-\Lambda_2^h)^T & r((Z^h)^T) & 0 & 0 & 0 \\
0 & 0 & d(I_{S+1}) & 0 & \Upsilon^T & 0 \\
r(\Delta u^h) & r((\Delta x^h)^T) & r((\Delta \lambda^h)^T) & \Delta p^T & \Delta \tau^T & 0
\end{pmatrix}
\]

where \(\Lambda, \Lambda_2,\) and \(Z^h\) were defined in the proof of Theorem 6. The submatrix \(\Upsilon^T\) is the \(H(S+1) \times I(S+1)\) matrix defined as follows:

\[
\Upsilon^T(j, k) = \begin{cases} 
1 & \text{if } h \in \mathcal{H}^*_{i} \text{ and } j = k + (h - i)(S + 1) \\
0 & \text{otherwise}
\end{cases}
\]

Lemma 2 \((\Delta u^h, \Delta p^T) \neq 0 \ \forall h \in \mathcal{H}\).

Proof. See Subsection 7.6.4. ■
From Lemmas 1 and 2, the first and last row blocks of $$\left( \begin{array}{c} \Psi^T \\ \nu^T \end{array} \right) D_A \Phi'$$ are linearly independent from the others. By the definition of $$\Lambda_2^h$$, the $$(H(S + 1) + G - (S + 1)) \times HG$$ submatrix

$$\begin{pmatrix} -P & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -P \end{pmatrix} - (\Lambda_2^h)^T \quad \begin{pmatrix} -\Lambda_2^h \end{pmatrix}^T$$

is a full rank matrix. The submatrix $$d(I_{S+1})$$ has full row rank. Thus, $$\left( \begin{array}{c} \Psi^T \\ \nu^T \end{array} \right) D_A \Phi'$$ has full rank (Eq. 22 has been met), finishing the argument in Case I.

### 7.6.2 Case II: $$(\Delta x^h)^T = 0$$ for some $$h \in \mathcal{H}$$

I will show that over a generic subset of $$\mathcal{E}$$, Eq. 21 has no solution. A subset of the equations $$\nu^T \Psi = 0$$, specifically corresponding to the columns of $$\Psi$$ for derivatives with respect to $$(x^h, \lambda^h)_{h \in \mathcal{H}}$$, is given by:

$$\begin{align*}
\Delta u^h D u^h(x^h) + (\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda &= 0. \quad (23.a) \\
- (\Delta x^h)^T P^T &= 0. \quad (23.b)
\end{align*}$$

Suppose $$\exists h' \in \mathcal{H}$$ such that $$(\Delta x^{h'})^T = 0$$. From Eq. (23.a) and $$\Phi(\xi, \theta) = 0$$, I obtain

$$\Delta u^{h'} D u^{h'}(x^{h'}) - (\Delta \lambda^{h'})^T P - \Delta p^T \Lambda = 0$$

$$Du^{h'}(x^{h'}) - \lambda^{h'} P = 0$$

which together imply that $$\Delta p^T = 0$$ and $$(\Delta \lambda^{h'})^T = \Delta u^{h'} \lambda^{h'}$$. For all other $$h \neq h'$$, postmultiply $$\Delta u^h D u^h(x^h)$$ by $$\Delta x^h$$ and use both first order conditions in $$\Phi$$ and Eq. (23.b) to get $$\Delta u^h D u^h(x^h) \Delta x^h = 0$$. Next, postmultiply Eq. (23.a) by $$\Delta x^h$$ and use the previous fact and Eq. (23.b) to arrive at $$(\Delta x^h)^T D^2 u^h(x^h) \Delta x^h = 0$$. By Assumption 3, $$(\Delta x^h)^T = 0 \forall h \in \mathcal{H}$$. Thus $$\forall h \in \mathcal{H}$$, $$(\Delta \lambda^h)^T = \Delta u^h \lambda^h$$.

From the equations $$\nu^T \Psi = 0$$ corresponding to the columns of $$\Psi$$ for derivatives with respect to $$(\tau^h)_{h \in \mathcal{H}}$$:

$$((\Delta \lambda^h)^T + \Delta \tau^i) = 0, \quad (24)$$
where \( h \in \mathcal{H}_i \). From Eq. 24, \( \forall i = 1, \ldots, I \), \( (\Delta \lambda^h)^T = (\Delta h^i)^T \) \( \forall h \in \mathcal{H}_i^* \). Thus, \( \forall i = 1, \ldots, I \),

\[
\Delta u^h \lambda^h = \Delta (h^i) \lambda^i \quad \forall h \in \mathcal{H}_i^*. 
\] (25)

The following is the equation from \( \nu^T \Psi = 0 \) corresponding to derivatives with respect to \( p \):

\[
\sum_{h \in \mathcal{H}} \Delta \lambda^h(s) \left( e^h_L(s) - x^h_L(s) \right)^T = 0 \quad \forall s \in \mathcal{S}. 
\] (26)

For the analysis to hold at this point, I must use Assumption 6: \( L > 1 \). From Eq. 26, only consider the first physical commodity, \( l = 1 \). Using the equality \( (\Delta \lambda^h)^T = \Delta u^h \lambda^h \) and Eq. 25, Eq. 26 simplifies to \( \sum_{i=1}^I \left[ \Delta u^h \lambda^i(s) \sum_{h \in \mathcal{H}_i^*} (e^h_i(s) - x^h_i(s))^T \right] = 0 \quad \forall s \in \mathcal{S} \), which can be written in the following matrix form:

\[
\begin{pmatrix}
\lambda^1(0) \sum_{h \in \mathcal{H}_i^*} (e^h_i(0) - x^h_i(0)) & \ldots & \lambda^h_i(0) \sum_{h \in \mathcal{H}_i^*} (e^h_i(0) - x^h_i(0)) \\
\vdots & \ddots & \vdots \\
\lambda^1(S) \sum_{h \in \mathcal{H}_i^*} (e^h_i(S) - x^h_i(S)) & \ldots & \lambda^h_i(S) \sum_{h \in \mathcal{H}_i^*} (e^h_i(S) - x^h_i(S))
\end{pmatrix}
\begin{pmatrix}
\Delta u^1 \\
\vdots \\
\Delta u^h_i
\end{pmatrix} = 0. 
\] (27)

By assumption in Theorem 7, \( I \leq S + 1 \).

**Lemma 3** Under Assumptions 1-6, over a generic subset of household endowments \( \mathcal{E} \) and provided \( I \leq S + 1 \), the matrix

\[
\begin{pmatrix}
\lambda^1(0) \sum_{h \in \mathcal{H}_i^*} (e^h_i(0) - x^h_i(0)) & \ldots & \lambda^h_i(0) \sum_{h \in \mathcal{H}_i^*} (e^h_i(0) - x^h_i(0)) \\
\vdots & \ddots & \vdots \\
\lambda^1(S) \sum_{h \in \mathcal{H}_i^*} (e^h_i(S) - x^h_i(S)) & \ldots & \lambda^h_i(S) \sum_{h \in \mathcal{H}_i^*} (e^h_i(S) - x^h_i(S))
\end{pmatrix}

has full column rank.

**Proof.** See Section 7.6.5. \( \blacksquare \)

From Lemma 3 and Eqs. 25 and 27, then \( \Delta u^h = 0 \quad \forall h \in \mathcal{H} \). This implies \( (\Delta \lambda^h)^T = 0 \) \( \forall h \in \mathcal{H} \) and \( \Delta T = 0 \) (using Eq. 24). The entire vector \( \nu^T = 0 \), which cannot be since \( \Phi' \) guarantees that \( \nu^T \nu / 2 = 1 \). I conclude that generically Case II is not possible. This completes the proof of Theorem 7.

### 7.6.3 Proof of Lemma 1

The set \( \mathcal{U} \) is infinite-dimensional and is endowed with the \( C^3 \) uniform convergence topology on compact sets. This means that a sequence of functions \( \{u^\nu\} \) converges uniformly to \( u \) iff \( \{Du^\nu\} \), \( \{D^2u^\nu\} \), and \( \{D^3u^\nu\} \) uniformly converge to \( Du, D^2u, \) and \( D^3u \), respectively.
Additionally, any subspace of $U$ is endowed with the subspace topology of the topology of $U$. I will use the regularity result from Theorem 6 to define utility functions as locally belonging to the finite-dimensional subset $A \subseteq U$.

Using Theorem 6, pick a regular value $\tilde{\theta}$. For that $\tilde{\theta}$, there exist finitely many equilibria $\bar{x}_j$, $j = 1, \ldots, J$. Further, there exist open sets $\Theta'$ and $A_j^{th}$ s.t. $\bar{x}_j^{th} \subseteq A_j^{th}$, the sets $A_j^{th}$ are disjoint across $j$, and $\forall \theta \in \Theta'$, $\exists!$ equilibrium $x_j^{th} \in A_j^{th}$. Choose $A_j^{th}$ such that the closure $\bar{A}_j^{th}$ is compact and there exist disjoint open sets $\tilde{A}_j^{th}$ s.t. $A_j^{th} \subset \tilde{A}_j^{th}$.

For each household, define a bump function $b^h : X^h \to [0, 1]$ with $J$ bumps as $b^h = 1$ on $\cup_j A_j^{th}$ and $b^h = 0$ on $(\cup_j \tilde{A}_j^{th})^c$. Now, I define $u^h$ in terms of a $G \times G$ symmetric matrix $A^h$ by:

$$u^h(x^h; A^h) = \bar{u}^h(x^h) + \frac{1}{2} b^h(x^h) \sum_j [(x^h - \bar{x}_j^{h})^T A^h (x^h - \bar{x}_j^{h})].$$

The space of symmetric matrices $(A^h)_{h \in \mathcal{H}} \in A$ is a finite dimensional subspace of $U$. Since $A$ has the subspace topology of $U$, then $u^h(\cdot; A^{h,v}) \to u^h(\cdot; A)$ iff $A^{h,v} \to A$. Taking derivatives with respect to $x^h \in A_j^{th}$ yields:

$$D_x u^h(x^h; A^h) = D\bar{u}^h(x^h) + A^h (x^h - \bar{x}_j^{h})$$

$$D_{xx} u^h(x^h; A^h) = D^2\bar{u}^h(x^h) + A^h.$$

$A$ is a $HG(G + 1)/2$ dimensional space, so write $A^h$ as the vector

$$\left((A^h_{j,j})_{j=1,\ldots,G}; (A^h_{j,k})_{j<k,j=1,\ldots,G-1}\right).$$

Postmultiply $D_{xx}^2$ by $\Delta x^h$:

$$D_{xx}^2 u^h(x^h; A^h) \Delta x^h = D^2\bar{u}^h(x^h) \Delta x^h + A^h \Delta x^h.$$

Taking derivatives with respect to the parameter $u^h$ is equivalent to taking derivatives with respect to $A^h$:

$$D_{A^h} \left(D_{xx}^2 u^h(x^h; A^h) \Delta x^h\right) = \left(\begin{array}{ccccccc}
\Delta x_1^h & 0 & 0 \\
0 & \ldots & 0 & \Sigma(1) & \ldots & \Sigma(G - 1) \\
0 & 0 & \Delta x_G^h
\end{array}\right) \in \mathbb{R}^{G(G + 1)/2}$$

41
where the submatrix $\Sigma(j)$ is defined as

$$\Sigma(j) = \begin{pmatrix} 0 \in \mathbb{R}^{j-1,G-j} \\
\Delta x^h_{j+1} & \ldots & \Delta x^h_G \\
\Delta x^h_j & 0 & 0 \\
0 & \ldots & 0 \\
0 & 0 & \Delta x^h_j \end{pmatrix} \in \mathbb{R}^{G,G-j}.$$ 

Thus, since $\Delta x^h \neq 0$ (without loss of generality $\Delta x^h_j \neq 0$), then

$$\text{rank} D_{A^h} \left( D^2_{x^h} u^h(x^h; A^h) \Delta x^h \right) = G. \quad (28)$$

The derivative $D_{A^h} \Psi' = \left( D_{A^h} \left( \Psi^T \nu \right) \right)$, where the only nonzero terms of $D_{A^h} \left( \Psi^T \nu \right)$ correspond to the rows for derivatives with respect to $x^h$, $D_{A^h} \left( D^2 u^h(x^h; A^h) \Delta x^h \right)$.\footnote{From the first derivative, $D_{x^h} u^h(x^h; A^h) = D\bar{u}(x^h) + A^h(x^h - \hat{x}^h) = D\bar{u}(x^h)$ evaluated at $\tau = \tau^*$ (since $x^h = \hat{x}^h$). Thus $D_{A^h} \left( D_{x^h} u^h(x^h; A^h) \Delta u^h \right) = 0$.}

Using Eq. 28, then

$$d \left( A^h \right) = \begin{bmatrix} D_{A^h} \left( D^2 u^1(x^1; A^1) \Delta x^1 \right) & 0 & 0 \\
0 & \ldots & 0 \\
0 & 0 & D_{A^h} \left( D^2 u^H(x^H; A^H) \Delta x^H \right) \end{bmatrix}$$

is a full row rank matrix of size $HG \times HG(G+1)/2$.

### 7.6.4 Proof of Lemma 2

Suppose not, that is $(\Delta u^h, \Delta p^T) = 0$ for some $h$. Then from the set of equations $\nu^T \Psi = 0$ corresponding to derivatives with respect to $x^h$, $(\Delta x^h)^T D^2 u^h(x^h) - (\Delta x^h) P$ (see Eq. (23.a)). From the set of equations $\nu^T \Psi = 0$ corresponding to derivatives with respect to $\lambda^h$, $(\Delta x^h)^T P^T = 0$ (see Eq. (23.b)). Thus, $(\Delta x^h)^T D^2 u^h(x^h) \Delta x^h = 0$ and from Assumption 3, $(\Delta x^h)^T = 0$, a contradiction of Case I.
7.6.5 Proof of Lemma 3

To prove this, first define

\[
Z^* = \left( \begin{array}{c}
\lambda^1(0) \sum_{h \in \mathcal{H}_1}(e^h(0) - x^h(0)) \\
\vdots \\
\lambda^1(S) \sum_{h \in \mathcal{H}_1}(e^h(S) - x^h(S)) \\
\vdots \\
\lambda^h(0) \sum_{h \in \mathcal{H}_1}(e^h(0) - x^h(0)) \\
\vdots \\
\lambda^h(S) \sum_{h \in \mathcal{H}_1}(e^h(S) - x^h(S)) \\
\end{array} \right).
\]

Since this proof is independent from the proof of Theorem 6, notation will be repeated. I will show that the matrix

\[
M = D_{\xi, \omega, e} \left( \begin{array}{c} 
\Phi(\xi, \theta) \\
Z^* \omega \\
\omega^T \omega / 2 - 1 
\end{array} \right)
\]

has full row rank. This suffices to prove that over a generic subset of \( \mathcal{E} \) the matrix \( Z^* \) has full column rank. To show that \( M \) has full row rank, premultiply by the row vector

\[
\nu^T = (\Delta x^T, \Delta \lambda^T, \Delta \gamma^T, \Delta \mu^T, \Delta p^T, \Delta z^T, \Delta \omega).
\]

The lemma is proved upon showing that \( \nu^T = 0 \). For convenience, the vector \( \nu^T \) is divided into the indicated subvectors which correspond sensibly with the following equations of

\[
\left( \begin{array}{c}
\Phi(\xi, \theta) \\
Z^* \omega \\
\omega^T \omega / 2 - 1 
\end{array} \right):
\]

\[
\Delta x^T \iff FOCx \\
\Delta \lambda^T \iff BC \\
\Delta \gamma^T \iff FOC\gamma \\
\Delta \mu^T \iff AC \\
\Delta p^T \iff MCx \\
\Delta z^T \iff Z^* \omega \\
\Delta \omega \iff \omega^T \omega / 2 - 1.
\]

I shall list the equations of \( \nu^T M = 0 \) in the order that is most convenient to obtain \( \nu^T = 0 \). At my disposal are \( \Phi(\xi, \theta) = 0 \) and \( \omega \neq 0 \).

As **Step One**, for the columns corresponding to derivatives with respect to \( (x^h)_{h \in \mathcal{H}} \) and
Further, over a generic subset of the columns corresponding to derivatives with respect to \( h \in \mathcal{H}_i \), and 

\[
\begin{align*}
(\Delta x^h)^T D^2 u^h(x^h) - (\Delta \lambda^h)^T P - \Delta p^T \Lambda - \Delta z^T \Lambda_5^h &= 0 \\
(\Delta \lambda^h)^T P + \Delta p^T \Lambda + \Delta z^T \Lambda_5^h &= 0
\end{align*}
\]

where the matrices \( P \) and \( \Lambda \) are as defined previously, \( \Lambda_5^h = \Lambda_5^{h^*} \) \( \forall h \in \mathcal{H}_i \), and \( \Lambda_5^h \) is the \((S + 1) \times G\) matrix

\[
\Lambda_5^h = \begin{pmatrix}
\lambda^{h^*}(0) \omega^{h^*} & \cdots & 0 \\
0 & \ddots & 0 \\
0 & 0 & \lambda^{h^*}(S) \omega^{h^*} & \cdots & 0
\end{pmatrix}.
\]

From the definitions of \( \Lambda \) and \( \Lambda_5^h \), \( (\Delta \lambda^h)^T = 0 \) \( \forall h \in \mathcal{H} \), \( \Delta p^{l,s} = 0 \) \( \forall (l, s) \notin \{(1, 0), \ldots, (1, S)\} \), and

\[
\Delta p^{1,s} + \Delta z^s \lambda^{h^*}(s) \omega^{h^*} = 0 \quad \forall s \in S \text{ and } \forall i = 1, \ldots, I.
\] (29)

Further, \( (\Delta x^h)^T D^2 u^h(x^h) \Delta x^h = 0 \) \( \forall h \in \mathcal{H} \). From Assumption 3, \( (\Delta x^h)^T = 0 \) \( \forall h \in \mathcal{H} \).

As **Step Two**, using the **second** step of the proof of Theorem 6 (Subsection 7.4), the columns corresponding to derivatives with respect to \( (\lambda^h)^{h \notin \{h_1^*, \ldots, h_I^*\}} \) and \( (\hat{\gamma}^h)^{h = h_1^*, \ldots, h_I^*} \) imply that \( (\Delta \hat{\gamma}^h)^T = 0 \) \( \forall h = h_1^*, \ldots, h_I^* \) and \( \Delta \mu^h = 0 \) \( \forall h \notin \{h_1^*, \ldots, h_I^*\} \). For any \( i = 1, \ldots, I \), consider the columns corresponding to derivatives with respect to \( \lambda^{h_i^*} \):

\[
\Delta z^T \Lambda_6^{h_i^*} = 0,
\] (30)

where \( \Lambda_6^{h_i^*} \) is the \((S + 1) \times (S + 1)\) matrix

\[
\Lambda_6^{h_i^*} = \begin{pmatrix}
\sum_{h \in \mathcal{H}_i^*}(e^h(0) - x^h(0)) \omega^{h_i^*} & 0 & 0 \\
0 & \ddots & 0 \\
0 & 0 & \sum_{h \in \mathcal{H}_i^*}(e^h(0) - x^h(0)) \omega^{h_i^*}
\end{pmatrix}.
\]

As Eq. 30 holds \( \forall i = 1, \ldots, I \), I obtain that

\[
\Delta z^s \omega^{h_i^*} \sum_{h \in \mathcal{H}_i^*}(e^h(0) - x^h(0)) = 0 \quad \forall i = 1, \ldots, I \text{ and } \forall s \in S.
\]

Over a generic subset of \( \mathcal{E} \) (Corollary 2), \( \sum_{h \in \mathcal{H}_i^*}(e^h(0) - x^h(0)) \neq 0 \) \( \forall s \in S \) and \( \forall i = 1, \ldots, I \). Since \( \omega \neq 0 \), then \( \forall s \in S \), there exists \( i \) such that \( \omega^{h_i^*} \sum_{h \in \mathcal{H}_i^*}(e^h(s) - x^h(s)) \neq 0 \). Thus \( \Delta z^s = 0 \) \( \forall s \in S \). From Eq. 29, the remaining terms of \( \Delta p^T \) are equal to 0, namely
\( \Delta p^{1,s} = 0 \) for \( s \in S \). From the columns corresponding to derivatives with respect to \( \omega \),
\( \Delta z^T Z^* + \Delta \omega (\omega)^T = 0 \). With \( \omega \neq 0 \), the scalar \( \Delta \omega = 0 \). Thus \( \nu^T = 0 \) and the proof of Lemma 3 is complete.

**References**


