Identification and Estimation of Online Price Competition with an Unknown Number of Firms*

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Abstract

This paper considers identification and estimation of a general model for online price competition. We show that when the number of competing firms is unknown, the underlying parameters of the model can still be identified and estimated employing recently developed results on measurement error. With the estimates of model parameters, we are able to analyze the competitive effects of online competition when the number of firms changes. We illustrate our methodology using UK data for personal digital assistants and employ the estimates to simulate competitive effects. Our results reveal that heightened competition has differential effects on the prices paid by different consumer segments.

Keywords: E-Retail Markets, Nonparametric Identification, Structural Estimation

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1 Introduction

Empirical competitive effects analysis is a fundamental tool for assessing the impact of market structure on consumer welfare. Such analysis can be controversial and challenging in even fairly simple brick-and-mortar retail environments. One major complication is that the relevant number of competitors in such markets is often unobserved by the econometrician. To the best of our knowledge, there are no ready tools available to empirically analyze and assess potential competitive effects involving online retailers. The absence of such tools or analyses stems, in part, from the fact that (1) online prices display considerable price dispersion, which substantially complicates predicting the price effects for a change of market structure; and (2) the number of (potential) competitors in the online channel is typically an unknown.

This paper represents a first attempt to econometrically estimate the competitive effects of changes in the number of firms in an online market when the number of sellers is unobserved. The model that we structurally estimate assumes (1) online firms are symmetric, pure-play e-retailers; (2) the number of (potential) online competitors at any point in time is known to firms but not to the econometrician; and (3) online buyers may be segmented into two types: price sensitive "shoppers," who rely on a price comparison site to find the best deal, and price insensitive "loyals," who simply visit their preferred online firm's website. This benchmark environment is the standard framework for modeling e-retail competition; see Baye, Morgan, and Scholten (2006) for a survey of this literature.

We first present a general model of online price competition that nests standard models ranging from Varian (1980) to Iyer, *et al.* (2005) as special cases. The model enriches existing models of online price competition, including Baye and Morgan (2001), by adding two realistic features: (1) firms pay platforms for clicks; and (2) not all clicks result in sales. In such a model, the observed price distribution represents a combination of the realized number of firms choosing to list on the site together with their realized prices, both of which are stochastic and depend on the unobserved number of actual competitors. Thus, it is essential to recover the price distribution conditional on the true number of competitors in order to estimate the model parameters and conduct competitive analysis. We show that, using the results from the recent econometric literature on misclassfication (e.g., Hu, 2008), this price distribution can be nonparametrically identified from the observed number of competitors and the listed prices. Based on the results of this identification, we present a two-step procedure of estimation, and demonstrate our procedure performs well using a simple Monte Carlo experiment. To illustrate our methodology, we structurally estimate the model based on UK data for personal digital assistants, and then use these estimates to simulate the competitive effects for changes in the number of competitors.

Our empirical results show that, at least in some instances, competitive effects in online markets are more similar to those predicted by the simple homogeneous product Bertrand model than might be expected given the price dispersion observed in (and predicted by theoretical models of) e-retail markets. However, there are potential distributional effects: if the number of firms decreases from three to two, the average transaction price paid by price sensitive "shoppers" increases by 2.88 percent, while the average transaction price paid by consumers "loyal" to a particular firm *declines* by 1.37 percent.[1]

We make two main contributions: (1) to present a methodology for identification and estimation of online retail markets in which the number of competitors is observed by firms but unknown to the econometrician; and (2) to provide some empirical evidence on how consumer welfare is affected by the number of competitors.

Most closely related to our work is An, Hu, and Shum (2010), which identifies and estimates a model of first-price auctions where the number of bidders is known to the auction participants but unobserved by the researcher. The present paper differs from An, Hu, and Shum (2010) by focusing on a completely different pricing environment as well as an application that highlights potential distributional effects of changes in competition in online markets. Furthermore, the present paper enriches the econometric methodology employed in An, Hu, and Shum (2010) by showing that the method works well for a modest-sized sample even though the estimation involves large dimensional matrices.[2]

We organize the rest of the paper as follows. In Section 2, we present a general model of online price competition. In Section 3, we show the nonparametric identification results. In Section 4, we describe a two-step estimation procedure for the proposed general model. In Section 5, we present an empirical application of our methodology using UK data for personal digital assistants. Section 6 concludes. The appendix contains miscellaneous proofs.

2 Model of Online Price Competition

To fix ideas, suppose the market consists of a commonly known number of firms ($N > 1$) that produce at a constant marginal cost of $m \geq 0$. Firms offer identical products for sale through their individual websites, which may have different characteristics or provide

[1] Armstrong (2008) points out that similar distributional effects are theoretically possible in the context of consumer protection policy, while Baye (2008) notes that this is a theoretical possibility in antitrust.

[2] We need to deal with 10×10 matrices while the matrices in An, Hu, and Shum (2010) is 3×3.

different types of service. Some consumers, who we call "loyals," value these services and purchase by directly visiting the website of their preferred firm. Other consumers, who we call "shoppers," care only about price. They first access a price comparison site to obtain a listing of the prices charged by sellers advertising at the site and click through to the firm offering the lowest price. If no prices are listed, they visit the website of a randomly selected firm.[3] All consumers have unit demand and a maximal willingness to pay of r.

It is widely recognized that conversion rates in online markets are low—only a fraction of consumers that click on a price at a comparison site follow through by making a purchase. To account for this, we assume that consumers are in the mood to buy with probability $\gamma \in (0, 1]$ and in the mood to merely "look" with probability $(1 - \gamma)$. Thus, γ may be interpreted as the conversion rate—the fraction of clicks that are converted into sales. Finally, we assume that each firm attracts $L \geq 0$ loyals and that there are a total of $S > 0$ shoppers.

We now turn to the details of firm behavior. To advertise at the comparison site, a firm must pay an (explicit or implicit) amount $\phi > 0$ to list its price, plus a cost per click (CPC) of $c \geq 0$ each time a consumer clicks on its price advertisement (listing). Thus, firm i's strategy consists of a continuous pricing decision (p_i) and a zero-one decision to advertise its price at the comparison site. Let α_i denote the probability that firm i chooses to advertise on the comparison site. A firm that does not advertise its price on the comparison site avoids paying listing and clickthrough fees, but at the potential cost of failing to attract the shoppers visiting the comparison site.

When platform fees are not too high, there is an active market for listings at the comparison site. For this case, we characterize the symmetric equilibrium pricing and advertising strategies of firms competing in this online environment (see Appendix A for a proof).

Proposition 1 *Suppose* $0 < \phi < S\left((r-m)\gamma\frac{N-1}{N} - c\right)$ *and* $0 \leq c < (r-m)\gamma\frac{N-1}{N}$. *Then in a symmetric Nash equilibrium:*

(a) Each firm lists its price on the comparison site with probability

$$\alpha^* = 1 - \left(\frac{\phi}{S\left((r-m)\gamma\frac{N-1}{N} - c\right)}\right)^{\frac{1}{N-1}} \in (0, 1)$$

(b) Conditional on listing a price at the comparison site, a firm's advertised price may be viewed as a random draw from

$$F^*(p) = \frac{1}{\alpha^*}\left(1 - \left(\frac{(r-p)\gamma L + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}\left((r-m)\gamma - c\right)}{S\left((p-m)\gamma - c\right)}\right)^{\frac{1}{N-1}}\right) \tag{1}$$

[3] See Proposition 1 in Baye and Morgan (2001) for the sorts of arguments required to ensure that this is an optimal decision rule for shoppers.

on $[p_0, r]$, *where* $p_0 = m + \frac{1}{(S\gamma + L\gamma)}\left(\gamma L(r-m) + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c) + Sc\right) \in (m, r)$.

(c) A firm that does not advertise on the comparison site charges a price of $p_i = r$ *on its own website.*

(d) Each firm earns expected profits of

$$E\pi = (r-m)\gamma L + \frac{\phi}{N\left(1 - \frac{c}{(r-m)\gamma}\right) - 1}$$

Notice that this model extends the original Baye and Morgan (2001) model to an environment in which all transactions take place online, and accounts for clickthrough fees as well as conversion rates that are potentially less than unity. Consistent with the empirical literature, the model implies that prices listed at the comparison site are necessarily dispersed in equilibrium, and that the number of firms actually listing prices at the comparison site on any given date is generally less than the total number of firms in the market.[4] This model nests a variety of other models as special cases, including Rosenthal (1980), Varian (1980), Narasimhan (1988), Iyer and Pazgal (2003), Baye, *et al.* (2004), and Iyer, *et al.* (2005). Unlike some of these special cases, the general model is flexible enough to allow differing competitive effects on consumer welfare.[5]

Under the maintained hypothesis that firms' listed prices are distributed according to equation (1), it is, in principle, possible to estimate the underlying parameters of the model. Unfortunately, data from price comparison sites reveal A, the realized number of firms choosing to list prices at the site at a given time, but not N, the total number of firms in the market. The model indicates that A is a binomially distributed random variable with parameters (α, N) whereas N is a constant. The extant literature mostly finesses this problem. For example, Baye, Morgan and Scholten (2006) as well as Moraga-Gonzalez and Wildenbeest (2008) use the number of observed prices as a proxy for N, in effect assuming that $N = A$. Hong and Shum (2006) assume that $N = +\infty$ in their identification of price dispersion models.

The problem of the unobservability of N presents econometric challenges, especially when it varies over time or across products. The next section offers an identification procedure that explicitly accommodates the unobservability of N.

[4] See Baye *et al.* (2006) for a survey of about twenty studies documenting price dispersion of 10 to 50 percent in online markets.

[5] For instance, the Rosenthal model implies that when there are two or more competitors, average prices paid by all consumers *rise* with the number of competing firms.

3 Nonparametric Identification

The above model of online price competition is essentially a low-bid auction in which the firm offering the lowest price secures the price sensitive shoppers when it lists on the comparison site. As such, we can adapt the techniques of econometric analysis of auction models to our setting. Specifically, in this section we show that the equilibrium distribution of prices in Proposition 1 (along with one additional but rather mild condition) implies the identification conditions for standard auctions pioneered by Hu (2008) and An, Hu and Shum (2010).

Following these authors, suppose the maximum number of (potential) firms is K, and is known to the econometrician. The actual number of firms ($N > 1$), which may vary, is common knowledge to the firms but unknown to the econometrician. For reasons that will become clear, consider only dates in which two or more firms listed prices, and let A denote the number of price listings on a given date. For these dates, randomly select one of the listed prices. Partition prices into $K - 1$ bins, and let Z denote a discretization of the randomly selected price. Thus, $Z = K - i$ means that the randomly selected price lies in the ith highest bin.

From the econometrician's point of view: (a) N, A, and Z share the same support $\{2, ..., K\}$; (b) r, m, ϕ, γ, L, and S are unknown parameters; and (c) N is unobservable or in dispute.[6] If we let $\theta \equiv (r, m, \phi, \gamma, L, S)$, then under the hypothesis that the price data at the comparison site are generated according to F^* in equation (1), we may write the underlying (undiscretized) distribution of prices as $F(p|N)$ and the associated density as $f(p|N)$.[7] The lemma below shows that the equilibrium density of listed prices is independent of A and Z:

Lemma 1 $f(p|N) = f(p|A, Z, N)$.

Proof. Follows directly from the fact that firms' prices are determined prior to their knowing realizations of A and Z. ■

Next, notice that, given the data and the model, conditional on the fact that at least two firms list prices the probability that exactly A firms list at the comparison site is

$$g(A|N) = \frac{\binom{N}{A} (\alpha)^A (1-\alpha)^{N-A}}{1 - (1-\alpha)^N - N\alpha (1-\alpha)^{N-1}} \text{ for all } N \geq A \geq 2$$

It immediately follows that

[6] In the application that follows, the cost-per-click (c) is data and hence is not included in the set of parameters to be estimated.

[7] To ease the notational burden, we have suppressed θ in this notation.

Lemma 2 $g\left(A|N\right) = g\left(A|Z, N\right)$

Lemma 1 implies that auxiliary variables A and Z only affect the equilibrium density of prices through the unobservable number of firms, N. Analogously, Lemma 2 implies that the instrument Z affects the number of listed prices only through N.

Let $h\left(p, A, Z\right)$ denote the observed joint density of p, A and Z. Let $\psi\left(N, Z\right)$ denote the joint density of N and Z, which is unobserved because N is unobserved. This specification allows for the possibility that the true number of firms N might vary across products and over time without placing parametric restrictions on the data-generating process in this respect. Now, the law of total probability implies the following relationship between the observed and latent densities:

$$\begin{aligned} h\left(p, A, Z\right) &= \sum_{N=2}^{K} f(p|N, A, Z) g(A|N, Z) \psi(N, Z) \\ &= \sum_{N=2}^{K} f(p|N) g(A|N) \psi(N, Z), \end{aligned} \tag{2}$$

where the second equality follows from Lemmas 1 and 2.

Define:

$$\begin{aligned} H_{p,A,Z} &= \left[h(p, A = i, Z = j)\right]_{i,j} \\ G_{A|N} &= \left[g\left(A = i|N = k\right)\right]_{i,k} \\ \Psi_{N,Z} &= \left[\psi\left(N = k, Z = j\right)\right]_{k,j} \end{aligned}$$

and

$$F_{p|N} = \begin{pmatrix} f(p|N = 2) & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & f(p|N = K) \end{pmatrix} \tag{3}$$

All of these are $K - 1$-dimensional square matrices. Then equation (2) may be written in matrix notation as:

$$H_{p,A,Z} = G_{A|N} F_{p|N} \Psi_{N,Z} \tag{4}$$

Next, consider the observed joint density of A and Z. Again, the law of total probability together with Lemma 2 implies that

$$b(A, Z) = \sum_{N=2}^{K} g(A|N) \psi(N, Z) \tag{5}$$

or, using matrix notation analogous to that above,

$$B_{A,Z} = G_{A|N} \Psi_{N,Z} \tag{6}$$

Identification requires that the following rank condition be satisfied:

Condition 1 Rank $(B_{A,Z}) = K - 1$.

Since both A and Z are observable, Condition 1 may be verified from the data. Equation (6) implies

$$\text{Rank}\,(B_{A,Z}) \leq \min\left\{\text{Rank}\left(G_{A|N}\right), \text{Rank}\,(\Psi_{N,Z})\right\}, \tag{7}$$

and hence, Condition 1 implies that $G_{A|N}$ and $\Psi_{N,Z}$ are invertible. This induces our key identifying equation:

$$H_{p,A,Z}\,(B_{A,Z})^{-1} = G_{A|N} F_{p|N} \left(G_{A|N}\right)^{-1} \tag{8}$$

The matrix on the left-hand side can be formed from the data. The right-hand side matrix represents an eigenvalue-eigenvector decomposition of the left-hand side matrix since $F_{p|N}$ is diagonal (cf. equation (3)). This representation allows us to estimate the unknown matrices $F_{p|N}$ and $G_{A|N}$.

The theory model implies:

Lemma 3 *The eigenvalue-eigenvector decomposition in equation (8) is unique.*

Proof. Since, for all N, the distribution of equilibrium prices contains a common interval in the neighborhood of r, it then follows that for any $i, j \in \mathcal{N}$, the set $\{(p) : f(p|N = i) \neq f(p|N = j)\}$ has nonzero Lebesgue measure whenever $i \neq j$. This immediately implies the uniqueness of the eigenvalue-eigenvector decomposition. ■

With Lemma 3 in hand, it then follows that an eigenvalue decomposition of the observed $H_{p,A,Z}\,(B_{A,Z})^{-1}$ matrix recovers the unknown $F_{p|N}$ and $G_{A|N}$ matrices. Here, $F_{p|N}$ is the diagonal matrix of eigenvalues, while $G_{A|N}$ is the corresponding matrix of eigenvectors. Of course, $F_{p|N}$ and $G_{A|N}$ are only identified up to a normalization and ordering of the columns of the eigenvector matrix $G_{A|N}$. There is a clear, appropriate choice for the normalization of the eigenvectors because each column of $G_{A|N}$ should add up to one. The model also implies a natural ordering for the columns of $G_{A|N}$, since in the model $A \leq N$ with probability one. This implies that for any $i < j \in \mathcal{N}$, $g\,(A = j|N = i) = 0$. In other words, $G_{A|N}$ is an upper-triangular matrix, which, since it is invertible, has non-zero diagonal entries, i.e. $f\,(A = i|N = i) > 0$ for all $i \in \mathcal{N}$.

Finally, having recovered $G_{A|N}$, from equation (6), we have

$$\Psi_{N,Z} = \left(G_{A|N}\right)^{-1} B_{A,Z}$$

and hence $\Psi_{N,Z}$ is also recovered. To summarize, we have shown:

Proposition 2 *Suppose Condition 1 holds. Then $F_{p|N}$, $G_{A|N}$ and $\Psi_{N,Z}$ are identified (with $F_{p|N}$ pointwise in p).*

Upon identifying the price distribution conditional on the "true" number of firms, $F_{p|N}$, equation (1) uniquely determines the model parameters $\theta \equiv (r, m, \phi, \gamma, L, S)$. The identification is constructive and it implies a two-step procedure of estimation, which will be presented in the next section.

4 Estimation of the Model

We now describe how one may use the identification argument to estimate the model, given data from a price comparison site. Let t index each set of price observations. For each t, we observe A_t, the number of firms choosing to list their prices at the comparison site. Let $p_{it},\ i = 1, \ldots, A_t$ denote the A_t listed prices of product i. Our estimation procedure accounts for the fact that N_t is known to the competing firms at time t but is, in effect, a random variable from the perspective of the econometrician. While we cannot recover the specific value of N_t pertaining to each set of prices at each point in time, we are able to recover its marginal distribution.

To estimate the vector of parameters θ, we use the following two-step estimation procedure: In the first step, we use our key equation (8) to nonparametrically estimate $G_{A|N}$. In the second step, based on the parametric form of $F(p|N;\theta)$ in equation (1) and the estimation in the first step, we recover the vector of parameters θ by MLE.

Step One We first describe how to use observable data on prices (p) and the number of listing firms (A) to estimate $G_{A|N}$. Our methodology closely parallels the approach taken in An, Hu and Shum (2010). While the key identification equation (2) is stated in terms of the joint density $h(p, A, Z)$, faster convergence is achieved if instead we take the expectation over all prices given (A, Z). Specifically, let $E[p|A, Z] = \int p \frac{h(p,A,Z)}{b(A,Z)} dp$, i.e. the expected price conditional on some realization A, Z. It then follows from equation (8) that

$$E[p|A, Z]\, b(A, Z) = \sum_{N=2}^{K} E[p|N] \times g(A|N)\psi(N, Z)$$

where $E[p|N] = \int pf\left(p|N\right)dp$.

Now define the matrices:

$$H_{Ep,N,Z} \equiv \left[E\left(p|A=i, Z=j\right) b(A=i, Z=j)\right]_{i,j}, \tag{9}$$

and

$$F_{Ep|N} \equiv \begin{pmatrix} E\left[p|N=2\right] & 0 & 0 \\ 0 & ... & 0 \\ 0 & 0 & E\left[p|N=K\right] \end{pmatrix}.$$

Then, we have

$$H_{Ep,A,Z} = G_{A|N} F_{Ep|N} \Psi_{N,Z}$$

which is analogous to equation (4). Similarly, we can obtain the estimating equation by postmultiplying both sides of this equation by $B_{A,Z}^{-1}$. This yields the analogous identification equation:

$$H_{Ep,A,Z}\left(B_{A,Z}\right)^{-1} = G_{A|N} F_{Ep|N}\left(G_{A|N}\right)^{-1} \tag{10}$$

Consequently,

$$G_{A|N} = \zeta\left(H_{Ep,A,Z}\left(B_{A,Z}\right)^{-1}\right),$$

where $\zeta\left(\cdot\right)$ denotes the mapping from a square matrix to its eigenvector matrix.[8] Following Hu (2008), we may estimate the relevant matrices using sample averages:

$$\widehat{G}_{A|N} \equiv \zeta\left(\widehat{H}_{Ep,A,Z}\left(\widehat{B}_{A,Z}\right)^{-1}\right), \tag{11}$$

where

$$\widehat{H}_{Ep,A,Z} = \left[\frac{1}{T}\sum_{t}\frac{1}{A_j}\sum_{i=1}^{A_j} p_{it}\mathbf{1}(A_t = A_j, Z_t = Z_k)\right]_{j,k}. \tag{12}$$

Finally, let $\mathbf{g}\left(A\right)$ be a vector of marginal probabilities over the number of listings and let $\mathbf{\Gamma}_N$ denote the vector of the unknown frequency distribution of N. Then

$$\mathbf{g}\left(A\right) = G_{A|N}\mathbf{\Gamma}_N$$

and we may estimate the unknown distribution $\mathbf{\Gamma}_N$ using the data as follows:

$$\widehat{\mathbf{\Gamma}}_N = \left(\widehat{G}_{A|N}\right)^{-1}\hat{\mathbf{g}}\left(A\right) \tag{13}$$

where $\hat{\mathbf{g}}\left(A\right)$ denotes the empirical frequency of the number of listings.

[8] Note that if the distribution of listed prices is such that the average price is monotonically ordered in N, then an analog of Lemma 3 holds for expected prices as well. This guarantees that ζ is a unique mapping.

Step Two In the first step, we obtained estimates of $G_{A|N}$ and Γ_N nonparametrically. In the second step, we combine these estimates with the equilibrium restrictions on the price distribution from Proposition 1 to obtain estimates of the model's structural parameters, θ.

Let $l(p, A; \theta)$ denote the joint density of prices and number of listings, and let $\Gamma(N)$ represent the unknown frequency distribution of N. In equilibrium, A and p are independent conditional on N. Thus, this density may be written as

$$\begin{aligned} l(p, A; \theta) &= \sum_{N=2}^{K} g(A|N) f(p|N; \theta) \Gamma(N) \\ &= e_A G_{A|N} F_{p|N;\theta} \mathbf{\Gamma}_N \end{aligned}$$

where $e_A = (0, 0, ..., 1, ..., 0)$ is a row vector where the 1 appears as the Ath element. Hence the likelihood function $\mathcal{L}$ for the t-th set of prices is

$$\begin{aligned} \mathcal{L} &= \prod_{i=1}^{A_t} l(p_i, A_t; \theta) \\ &= \prod_{i=1}^{A_t} e_{A_t} G_{A_t|N} F_{p_i|N;\theta} \mathbf{\Gamma}_N \end{aligned}$$

Using the first step estimates, we can write this as

$$\begin{aligned} \ln \mathcal{L} &= \ln \widehat{l}(p_i, A_t; \theta) \\ &= \sum_{i=1}^{A_t} \ln \left(e_{A_t} \widehat{G}_{A_t|N} F_{p_i|N;\theta} \widehat{\mathbf{\Gamma}}_N \right) \end{aligned} \tag{14}$$

where $F_{p_i|N;\theta}$ is a diagonal matrix with diagonal element $f(p|N;\theta)$. From equation (1), it may be shown that the density associated with $F^*(p)$ is given by

$$\begin{aligned} f^*(p|N;\theta) &= \frac{1}{N-1} \left(F^*(p|N;\theta) - \frac{1}{\alpha^*} \right) \\ &\quad \times \left(\frac{\gamma L}{(r-p)\gamma L + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c)} - \frac{\gamma}{(p-m)\gamma - c} \right) \end{aligned}$$

for $p \in [p_0, r]$ and zero otherwise. Note that $\widehat{G}_{A_t|N}$ and $\widehat{\mathbf{\Gamma}}_N$ are estimated using the data, whereas $F_{p_i|N;\theta}$ is based on the theory model (we have added θ to the subscript of $F_{p_i|N}$ to emphasize the dependence on θ, which will be selected so as to maximize the likelihood function).[9]

[9]We do not estimate c because we have data on clickthrough fees, as discussed below.

5 Empirical Application

We apply the two-step estimation procedure to UK online price data obtained from Kelkoo.com for firms selling personal digital assistants. These data, which are described in detail in Baye *et al.* (2009), include the daily transactions prices (inclusive of taxes and shipping) charged by firms selling 18 models of personal digital assistants (PDAs) over the period from 18 September 2003 through 6 January 2004. During this period, an average of four firms sold each product at the comparison site, so on the surface this market might appear to be fairly concentrated. Table 1 presents some statistics on the data used in our analysis. Our data consists of $1,591$ product-dates. For $1,229$ of these product-dates, two or more firms listed prices, and we use these data in the estimation. Our estimation is based on clickthrough fees at Kelkoo.com of $c = .20$ (20 pence per click).

Since our estimation procedure requires a large number of observations, we pool across all 18 products in both the first and second step of our estimation procedure to estimate a common parameter vector, θ. Owing to a paucity of observations where the number of listings exceeds 10, we combine observations where more than 10 firms list prices into a single bin.[10] Correspondingly, both A and N take ten distinct values from $\{2, 3, ..., 10, 10+\}$. We use a discretized second listed price as the instrument Z, which has the same support as that of A and N. Hence, $G_{A|N}$ is a 10×10 matrix for purposes of estimation, with the first 9 columns corresponding to $N = 2, ...10$ and the last bin corresponding to $N > 10$.[11]

We observe that $A \leq N$ implies that the matrix on the right-hand side of the equation (10) should be upper triangular, and consequently the matrix on the left-hand side, $H_{Ep,A,Z}B^{-1}_{A,Z}$ should also be upper-triangular. However, the matrix $H_{Ep,A,Z}B^{-1}_{A,Z}$ is observed from the data and it may not be upper-triangular. An, Hu, and Shum (2010) argue that if we do not impose upper-triangularity on $H_{Ep,A,Z}B^{-1}_{A,Z}$ in the first step of estimation while constraining the estimated matrix $\widehat{G}_{A|N}$ to be upper-triangular in the second step of estimation, the asymptotic consistency and convergence properties of $\widehat{G}_{A|N}$ will not be affected since $H_{Ep,N,Z}B^{-1}_{A,Z}$ is upper-triangular asymptotically. We follow this procedure in the first step of our estimation.

An important issue in the second step of estimation is that likelihood function in equation (14) depends on an estimated $G_{A|N}$ matrix that uses only observations where two or more firms listed prices. Lemma 1, however, implies that $f(p|N, A \geq 2; \theta) = f(p|N; \theta)$; thus, our

[10] The maximum number of listings observed is 15.

[11] Baye and Morgan (2009) show that in an analogous model, the equilibrium price distribution as $N \to \infty$ is similar to that for finite values of N near the lower end of this last bin.

estimates of θ remain consistent even with this restriction.

5.1 Estimation Results

Table 2 reports the results of the first-stage estimation, estimate of the matrix $G_{A|N}$. The element $\widehat{G}_{A=i|N=j}(j \geq i)$ corresponds to the estimated probability (in the data used) that there are A firms listing prices on the comparison site when the population of firms is N. Although the estimation procedure places no constraints requiring that the resulting estimates are well-defined probabilities, Table 2 reveals that the resulting estimates do, in fact, have this property.

We now turn to the step 2 results. Recall that c is known data and not a parameter to be estimated. Following Baye and Morgan (2001), we set $L \equiv M/N$, so that M (the parameter to be estimated) represents the total number of loyal consumers in the market, and L is the (unobserved) number of loyals per firm on a given product-date. The resulting parameter estimates, along with bootstrapped standard errors, are reported in Table 3. The monetary parameters (r, m and ϕ) are denominated in GBP. As the table reveals, all of the parameters are precisely estimated.

To further examine the robustness of these estimates, we performed a Monte Carlo simulation, described in detail in Appendix B. Table B1 presents the results of this analysis. As that table shows, despite our modest sample size and the consequent need to pool over certain values of N and A, the procedure works well at recovering the deep structural parameters of the model with simulated data.

The parameter estimates in Table 3 indicate that, on an average day, a total of $M = 26.04$ consumers in the UK who are loyal to some online firm were interested in purchasing a PDA online, while $S = 13.16$ consumers were interested in purchasing online from the firm charging the best price. These estimates imply that about 34 percent of consumers in this online market are price-sensitive shoppers, while 66 percent are loyals. It is interesting to contrast our estimates with those of Brynjolfsson, Montgomery, and Smith (2003), who find that around 13% of consumers in US e-retail markets are shoppers. Given the relatively less-developed state of e-retail in the UK compared to the US at the time our data was collected, it is not altogether surprising to find that fewer UK customers had become "attached" to a particular online retailer.

The estimated conversion rate, $\gamma = .15$, implies that a firm listing on Kelkoo.com has to receive, on average, 6.67 clicks in order to generate one sale. At a cost of 20 pence per click, this translates into an average cost per sale of 1.33 GBP in addition to the fixed listing fee of

$\phi = 4.88$ GBP. Finally, notice that the estimated monopoly markup for a PDA, $(r - m)/m$, is about 66 percent.

5.2 Competitive Effects Analysis

The econometric framework described above, along with the structural estimates of the model of online price competition, permits us to address a number of issues that arise in the evaluation of the competitive effects when the number of online firms changes.

To accomplish this, we first substitute the parameter estimates reported in Table 3 into the expressions summarizing equilibrium behavior in Proposition 1; below we use carets to denote the resulting estimates. Next, we calculate the implied average prices conditional on a given number of firms and display them in Table 4. Column (a) in Table 4 indicates the total number of firms in the relevant market (N), which is unknown. Column (b) provides the estimated average price listed at the comparison site conditional on different numbers of competitors, where the average listed price is

$$E[p] = \int_{\widehat{p_0}}^{\widehat{r}} p d\widehat{F^*}(p).$$

As would be expected, Table 4 shows that the estimated average listed price declines as the number of firms increases—rather abruptly as one moves from monopoly to a duopoly, and modestly thereafter. Column (c) reports the estimated average minimum listed price, which is given by

$$E[p_{\min}] = \frac{1}{1 - \left(1 - \widehat{\alpha^*}\right)^N} \sum_{A=1}^{N} \binom{N}{A} \left(\widehat{\alpha^*}\right)^A \left(1 - \widehat{\alpha^*}\right)^{N-A} \int_{\widehat{p_0}}^{\widehat{r}} pA \left(1 - \widehat{F^*}(p)\right)^{A-1} d\widehat{F^*}(p)$$

Notice that this calculation takes into account the effect of a change in N on the equilibrium distribution of prices, firms' propensities to advertise prices at the comparison site, and the impact of a larger number of listings on the minimum order statistic. Accounting for this, Column (c) of Table 4 shows that the estimated average minimum listed price also declines as the number of firms increases.

While it might be tempting to base competitive effects analysis on these average prices (presuming the average prices are relevant for loyals and the average minimum prices are relevant for shoppers), this would be incorrect: Neither of these averages represents average transaction prices. To calculate the average transaction price paid by loyals, one needs to account for a firm's propensity to list prices on the comparison site. In particular, when a firm does not list on the comparison site, it charges the monopoly price at its own website. Thus, the average transaction price paid by a loyal customer is

$$E\left[p^L\right] = \widehat{\alpha^*} E\left[p\right] + \left(1 - \widehat{\alpha^*}\right)\widehat{r}.$$

Column (d) of Table 4 reports the estimated average transaction prices of loyal consumers. Notice that it declines abruptly as one moves from monopoly to duopoly, but then rises as the number of firms increases further.

Likewise, the average transaction price for shoppers must also account for listing decisions: The average transaction price paid by a price-sensitive shopper is given by

$$E\left[p^S\right] = \left(1 - \left(1 - \widehat{\alpha^*}\right)^N\right) E\left[p_{\min}\right] + \left(1 - \widehat{\alpha^*}\right)^N \widehat{r}$$

Column (e) of Table 4 reports the estimated average transaction price of shoppers, which declines as the number of firms increases.

Columns (d) and (e) highlight that shoppers and loyals are impacted differently by heightened competition: So long as there are at least two firms in the market, loyal consumers are harmed by heightened competition, while shoppers are unambiguously made better off by increased competition. The overall transaction price, reported in Column (f) of Table 4, is merely a weighted average of the shoppers' and loyals' estimated transaction prices, where the weighting factor is determined by the estimated fraction of consumers who are shoppers and loyals:

$$E\left[p^T\right] = \frac{\widehat{M}}{\widehat{S} + \widehat{M}} E\left[p^L\right] + \frac{\widehat{S}}{\widehat{S} + \widehat{M}} E\left[p^S\right]$$

In summary, the estimates in Table 4 reveal that the average listed price and the average minimum listed price both decline as the number of firms declines. This is consistent with standard reasoning, which suggests that heightened competition leads to lower prices. However, this ignores the endogenous listing decisions of firms, which is, of course, relevant for the *transaction* prices paid by consumers. Here, a more subtle story emerges. Both shoppers and loyals pay lower average transaction prices as the online market moves from monopoly to duopoly. Thereafter, the effects of increased competition diverge: Loyal consumers are harmed (pay higher average transaction prices) as the number of firms further increases, while shoppers benefit from heightened competition.

Table 5 uses the results in Table 4 to compute the competitive effects when the number of firms declines N to $N-1$, where column (a) represents the post-decline number of firms. Obviously, the direction of the price changes is identical to that in Table 4, but it is instructive to examine the implied percentage changes in prices to highlight the potential value of our methodology. So long as there is more than one firm in the market, a change of market structure will not harm the "average" online consumer. This conclusion is based on the assumption that firms in the online channel do not compete against firms in other channels. In effect, column (f) reveals that—even though models of online competition are more complex than standard homogenous product Bertrand competition and the "law of one price" does not hold online—the conclusions based on our estimates are similar to what one would have concluded based on the simple Bertrand model, at least in this particular online market: There are no adverse competitive effects of a change of market structure in this online market so long as there are two firms in the market.

6 Conclusions

We have shown that the econometric methodology from auctions may be used to identify and structurally estimate standard models of online competition– even when the number of competing firms is unobserved. The estimates can be employed to analyze the competitive effects induced by the change of number of firms. Our empirical results suggest that: (1) Online markets are less vulnerable to adverse competitive effects from reductions in the number of competing firms than one might expect given the plethora of papers documenting significant price dispersion in online markets; and (2) reductions in the number of competitors in online retail markets harm price sensitive shoppers but benefit customers who are loyal to a particular firm. We stress, however, that these findings are based on data from one e-retail market in the UK. While the model and econometric techniques developed in this paper are useful more generally, one should tread cautiously in generalizing too much from these data.

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A Proof of Proposition 1

As in Baye and Morgan (2001), it is readily seen that equilibrium has the following two key properties: (1) A firm must be indifferent between listing its price at the clearinghouse or not; and (2) a firm must earn the same expected payoff from posting any price $p \in [p_0, r]$ at the clearinghouse.

A firm that eschews the comparison site earns profits of

$$\pi_0 = (r-m)\,\gamma L + (r-m)\,(1-\alpha)^{N-1}\,\frac{\gamma S}{N} \tag{15}$$

A firm that advertises a price r on the site earns

$$\pi = (r-m)\,\gamma L + (r-m)\,(1-\alpha)^{N-1}\,\gamma S - c\,(1-\alpha)^{N-1}\,S - \phi$$

Since firms must be indifferent between listing or not, it then follows that $\pi = \pi_0$. We may use this equality to obtain a closed-form expression for α:

$$(r-m)\,\gamma L + (r-m)\,(1-\alpha)^{N-1}\,\frac{\gamma S}{N} = (r-m)\,\gamma L + (r-m)\,(1-\alpha)^{N-1}\,\gamma S - c\,(1-\alpha)^{N-1}\,S - \phi$$

Simplifying, this reduces to

$$\phi = (1-\alpha)^{N-1}\,S\left((r-m)\,\gamma\frac{N-1}{N} - c\right)$$

Or equivalently,

$$\begin{aligned}(1-\alpha)^{N-1} &= \frac{\phi}{S\left((r-m)\,\gamma\frac{N-1}{N} - c\right)} \\ &= \frac{N\phi}{S\left((r-m)\,\gamma\,(N-1) - Nc\right)}\end{aligned}$$

Hence, the equilibrium advertising propensity is:

$$\alpha^* = 1 - \left(\frac{\phi}{S\left((r-m)\,\gamma\frac{N-1}{N} - c\right)}\right)^{\frac{1}{N-1}} \tag{16}$$

The conditions on ϕ and c identified in Proposition 1 imply that $\alpha^* \in (0,1)$.

Substituting for α^* in equation (15) , we obtain equilibrium profits of :

$$\begin{aligned}\pi_0 &= (r-m)\,\gamma L + (r-m)\,\frac{\phi}{S\left((r-m)\,\gamma\frac{N-1}{N} - c\right)}\,\frac{\gamma S}{N} \\ &= (r-m)\,\gamma L + \frac{\phi}{N\left(1 - \frac{c}{(r-m)\gamma}\right) - 1}\end{aligned}$$

It remains to determine the equilibrium distribution of listed prices. Recall that a firm listing a price p, earns expected profits of

$$\pi(p) = (p-m)\gamma L + (p-m)(1-\alpha F(p))^{N-1}\gamma S - c(1-\alpha F(p))^{N-1} S - \phi$$

Such a firm must be indifferent between charging p and not advertising at all, i.e. $\pi(p) = \pi_0$. It is convenient to express π_0 in terms of α for the moment. Hence, we have:

$$\begin{aligned}\pi(p) &= (p-m)\gamma L + (p-m)(1-\alpha F(p))^{N-1}\gamma S - c(1-\alpha F(p))^{N-1} S - \phi \\ &= (r-m)\gamma L + (r-m)(1-\alpha)^{N-1}\frac{\gamma S}{N} = \pi_0\end{aligned}$$

Solving this expression for $(1-\alpha F(p))^{N-1}$, we obtain

$$\begin{aligned}(1-\alpha F(p))^{N-1} &= \frac{(r-p)\gamma L + (r-m)(1-\alpha)^{N-1}\frac{\gamma S}{N} + \phi}{S((p-m)\gamma - c)} \\ &= \frac{(r-p)\gamma L + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c)}{S((p-m)\gamma - c)}\end{aligned}$$

which implies

$$F(p) = \frac{1}{\alpha}\left(1 - \left(\frac{(r-p)\gamma L + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c)}{S((p-m)\gamma - c)}\right)^{\frac{1}{N-1}}\right)$$

To verify that $F(p)$ is a well-defined atomless probability distribution, we will first show that $F(r) = 1$, or equivalently, $(1-\alpha F(r))^{N-1} = (1-\alpha)^{N-1}$. To see this, note that

$$\begin{aligned}(1-\alpha F(r))^{N-1} &= \frac{\frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c)}{S((r-m)\gamma - c)} \\ &= \frac{N\phi}{S((r-m)\gamma(N-1) - Nc)} \\ &= (1-\alpha)^{N-1}\end{aligned}$$

where α is defined in equation (16).

Next, we determine the lower support of the equilibrium listed price distribution; that is p_0, where $F(p_0) = 0$. Equivalently, p_0 satisfies $(1-\alpha F(p_0))^{N-1} = 1$, or

$$\frac{(r-p_0)\gamma L + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c)}{S((p_0-m)\gamma - c)} = 1$$

Cross-multiplying and collecting the p_0 terms:

$$\gamma L r + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c) + S\gamma m + Sc = p_0(S\gamma + L\gamma)$$

Solving for p_0 gives

$$p_0 = m + \frac{1}{(S\gamma + L\gamma)}\left(\gamma L(r-m) + \frac{N\phi}{((r-m)\gamma(N-1) - Nc)}((r-m)\gamma - c) + Sc\right)$$

which exceeds m.

Finally, we verify that F is strictly increasing, or equivalently, that $(1-\alpha F(p))^{N-1}$ is strictly decreasing in p. Recall that

$$(1-\alpha F(p))^{N-1} = \frac{(r-p)\gamma L + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c)}{S((p-m)\gamma - c)}$$

and define $num \equiv (r-p)\gamma L + \frac{N\phi}{((r-m)\gamma(N-1)-Nc)}((r-m)\gamma - c) > 0$ and $den \equiv S((p-m)\gamma - c) > 0$. Differentiating with respect to p reveals

$$\frac{\partial(1-\alpha F(p))^{N-1}}{\partial p} = -\frac{\gamma L(den) + S\gamma(num)}{(den)^2} < 0$$

B Simulation

We present a simulation study demonstrating that our estimation procedure performs well in a controlled, small-sample environment that mirrors that analyzed in the empirical application. Based on the designed "true" parameter values $\left(\theta^{True}\right)$, we construct a simulated dataset based on the underlying theoretical model as follows.

For each simulated period, t, we randomly draw a number of firms for that period, $N_t \in \{2, 3, ..., 15\}$ from a discrete uniform distribution. Notice that the upper bound of this distribution corresponds to the maximum number of listings we observed across all product-dates in the actual data. Next, we make N_t Bernoulli draws with parameter $\alpha^*(N_t; \theta^{True})$ (defined in Proposition 1) to simulate whether each of these N_t firms listed or not. Let A_t denote the number of firms listing prices in simulated period t. For each of these A_t firms, we next draw a listed price from the distribution $F^*(p|N_t; \theta^{True})$ defined in Proposition 1. Following the estimation procedure in the paper, we retain data for this simulated period only if $A_t \geq 2$. We repeat this process until we have retained exactly $1,229$ simulated periods—the sample-size used in our actual estimation in the text.

Next, following the approach in the paper, we pool simulated observations where $A_t \geq 11$ into a single bin, and use the simulated data to estimate the model via our two-step estimation procedure. These estimates are presented in Table B1, along with standard errors obtained via bootstrapping (with 200 resamples employed). As Table B1 reveals, the parameters are precisely estimated, and very close to the true values.

Table 1: Summary Statistics*

Number of Listings (A)	# Observations	Percentage	Avg. price
1	362	22.75	319.49
2	357	22.44	304.04
3	155	9.74	306.54
4	161	10.12	342.14
5	141	8.86	329.35
6	110	6.91	308.66
7	113	7.10	308.35
8	76	4.78	304.83
9	35	2.20	290.70
10	35	2.20	280.35
> 10	46	2.89	281.87

* The data with number of listings $A = 1$ are not used.

Table 2: Estimated $G_{A|N}$ Matrix

	Number of firms (N)									
Number of listings(A)	2	3	4	5	6	7	8	9	10	> 10
2	1.0000	0.9068	0.7141	0.7526	0.6775	0.3143	0.7010	0.6528	0.2585	0.2532
3	0	0.0932	0.2358	0.1311	0.1180	0.1425	0.0458	0.0427	0.1245	0.1220
4	0	0	0.0501	0.0459	0.0413	0.1293	0.0002	0.0002	0.1794	0.1713
5	0	0	0	0.0704	0.0634	0.1517	0.1172	0.1092	0.1381	0.1351
6	0	0	0	0	0.0998	0.1243	0.0225	0.0209	0.1040	0.1018
7	0	0	0	0	0	0.1380	0.0939	0.0874	0.1077	0.1055
8	0	0	0	0	0	0	0.0194	0.0181	0.0419	0.0411
9	0	0	0	0	0	0	0	0.0687	0.0279	0.0273
10	0	0	0	0	0	0	0	0	0.0225	0.0221
> 10	0	0	0	0	0	0	0	0	0	0.0205

Table 3: Parameter Estimates and Bootstrapped Standard Errors*

Parameter	Estimate	Standard Error
ϕ	4.88	0.109
r	415.26	85.097
m	250.09	17.651
M	26.04	1.742
S	13.16	0.574
γ	0.15	0.004

* Standard errors calculated using bootstrap resampling (200 times).

Table 4: Estimated Transaction Prices

Number of Firms	Estimated Avg. Listed Price	Estimated Avg. Minimum Listed Price	Estimated Avg. Transaction Price Loyals	Estimated Avg. Transaction Price Shoppers	Estimated Avg. Transaction Price
(a)	(b)	(c)	(d)	(e)	(f)
1	415.26	415.26	415.26	415.26	415.26
2	366.57	354.60	368.06	354.65	363.56
3	365.71	344.49	373.19	344.73	363.64
4	363.23	336.43	377.40	336.87	363.79
5	360.46	329.93	380.79	330.53	363.92
6	357.77	324.59	383.56	325.33	364.01
7	355.25	320.11	385.87	320.96	364.08
8	352.92	316.29	387.84	317.25	364.14
9	350.76	312.99	389.53	314.03	364.18
10	348.77	310.10	391.00	311.23	364.22
11	346.93	307.56	392.30	308.75	364.25
12	345.22	305.30	393.45	307.54	364.27
13	343.63	303.26	394.48	304.57	364.30
14	342.14	301.43	395.41	302.78	364.31
15	340.75	299.77	396.25	301.16	364.33

Table 5: Percentage Change of Transaction Prices

Number of Firms	Estimated Change in Average Listed Price	Estimated Change in Avg. Minimum Listed Price	Estimated Change in Average Transaction Price Loyals	Estimated Change in Average Transaction Price Shoppers	Estimated Change in Average Transaction Price
(a)	(b)	(c)	(d)	(e)	(f)
1	13.28%	17.11%	12.82%	17.09%	14.22%
2	0.24	2.93	-1.37	2.88	-0.02
3	0.68	2.40	-1.12	2.34	-0.04
4	0.77	1.97	-0.89	1.92	-0.03
5	0.75	1.65	-0.72	1.60	-0.03
6	0.71	1.40	-0.60	1.36	-0.02
7	0.66	1.21	-0.51	1.17	-0.02
8	0.61	1.05	-0.43	1.02	-0.01
9	0.57	0.93	-0.38	0.90	-0.01
10	0.53	0.83	-0.33	0.80	-0.01
11	0.50	0.74	-0.29	0.72	-0.01
12	0.46	0.67	-0.26	0.65	-0.01
13	0.43	0.61	-0.24	0.59	-0.01
14	0.41	0.56	-0.21	0.54	-0.00

Table B1: Simulation Parameter Estimates and Standard Errors

Parameter	"True" value	Estimate	Standard Error
ϕ	4.88	4.10	1.831
r	415.26	421.68	25.868
m	250.09	258.94	46.814
M	26.04	25.33	3.236
S	13.16	9.20	5.205
γ	0.15	0.14	0.081