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The Parking Lot Problem

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Abstract

If there is competition for access to an underpriced good such as a free parking spot, the competition can eat up the entire surplus, eliminating the social value of the good. There is a discontinuity in social welfare between "enough" and "not enough," with the minimum social welfare being at slightly too small a parking lot because of the rent-seeking efforts of drivers. Uncertainty over the number of drivers actually increases social welfare if the parking lot size is set too small; if it is set optimally, the parking lot size will be well in excess of mean demand.

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1. Introduction

Finding a parking space is a perennial source of frustration. College campuses are especially prone to parking shortages. Tough competition between students for a limited number of parking spaces compels students to arrive well in advance their preferred time. The October 15, 1994 *Los Angeles Times* reports that in one California college, a parking permit is no more than a hunting license. "Survival of the earliest" is the rule, and students arrive hours in advance to find a parking spot. On the other hand, shopping malls display an apparently useless excess of parking spaces. In contrast to such public institutions as state universities and downtown retailing districts, which have too little parking, malls seem to go to ridiculous extremes with their acres of parking lots which are never full except around Christmas.

Parking studies recognize the importance of planning for a sufficient 'cushion' in excess of the necessary spaces. However, these studies are mostly concerned with the smooth flow of vehicles in and out of the parking area and unforeseen circumstances such as minor construction. The 1987 study by *Walker Parking Consultants* (commissioned by Indianapolis to develop a Regional Center Parking Plan) says, "Thus, a supply of parking operates at peak efficiency when occupancy is 85% to 90%. When occupancy exceeds this level, there may be delays and frustration in finding a space. The parking supply may be perceived as inadequate even though there are spaces available in the system" (http://www.bts.gov/NTL/DOCS/rc.html). We suggest that an equally important concern is strategic behavior by drivers. If the parking lot is built just slightly too small, rent-seeking competition can dissipate all of the rents from parking there, eliminating every trace of the lot's social usefulness and making its construction cost entirely wasteful. Hence, more important than any engineering concern is that a parking lot should be sufficiently large to avoid wasteful competition for parking spaces. It can easily be socially optimal to have parking lots that on average are half empty, as we will show below by example. Shopping malls seem to realize this better than public or nonprofit institutions.

It is natural to view empty spaces as a reason to reduce the amount of parking. Based on the finding that the average parking supply count exceeded demand count by 30%, for example, a Seattle report recommends reducing the parking supply there. The authors admit, however, that *"A major policy-related issue is how much allowance to provide over the design-level demand in setting the size of a given parking facility."*[1]

We believe that strategic behavior must be considered when deciding on parking lot size. Having 30% of the spaces empty may well indicate too *small* a parking lot size, given uncertainty in the demand for parking. We will show that there is a strategic asymmetry in the welfare effects of over- and under-supply of parking. Extra parking spaces are costly in proportion to their number, but even a small shortage in parking can result in a discontinuous and huge social loss, and the welfare loss can be higher at a small shortage than at a large shortage.

We thus construct a model of competition between drivers for parking spaces. The drivers face a trade-off between the disutility of arriving earlier than their preferred time and the increased probability of securing a space in the parking lot. Since the cost of arriving early is incurred regardless of whether a driver is successful in finding a parking space, the contest is a multi-unit all-pay auction. Due to the dynamic nature of the players' decisions on *when* to arrive at the lot, their strategies can be quite complex. Nevertheless, we are able to show that full (or almost full) rent dissipation occurs whenever the size of the lot is too small.

The model describes a dynamic competition for a given supply of goods. The trade-off between arriving later and reducing one's chances of obtaining a good is by no means limited to the parking problem. The trade-off can arise in a number of diverse applications. People in need of vaccination may try to overcome vaccine shortages by demanding the vaccine earlier, when there is still uncertainty about the severity and features of the next virus outbreak. Parents may sign up their kids to a daycare just to keep the spot to themselves when they really need it. Firms may preempt other firms by approaching potential acquisition targets before other firms. Bargain hunters may have to arrive earlier to ensure the availability of a product at a bargain price. People who have relatives living in different time zones may start calling their relatives ahead of their preferred time (at midnight of the New Year Eve) for fear of not being able to get through later.

Section 2 of the paper will provide a brief discussion of the literature on the subject. In Section 3, we describe parking as a dynamic discrete-time competition between a known

[1] See the *1991 Parking Utilization Study* undertaken by the Research and Market Strategy Division of the Transit Department in the Municipality of Metropolitan Seattle.

number of drivers for a limited number of parking spaces. We derive pure-strategy subgame-perfect equilibria under full observability of the parking lot, and show that such equilibria do not exist under unobservability. Section 4 establishes the full rent dissipation result for any subgame-perfect equilibria of the parking game. In Section 5, the optimal capacity is determined for a certain and uncertain number of drivers. We show that in the case of a known number of drivers, the optimal size of the parking lot is equal to the number of drivers. When the size of the parking lot is set before the uncertainty about the number of drivers is resolved and the planners only know its distribution, they should build a sufficiently large parking lot. On average, a large proportion of the parking spots will be unoccupied for a parking lot of an optimal size. Section 6 discusses the results and extensions of the model, and Section 7 concludes.

2. The Literature

The problem of managing a transportation system has been analyzed by urban economists and operations researchers. Queueing models with tolls, e.g. Naor (1969), assume an exogenous stationary customer arrival process and random service times. Customers benefit from the service, but have to incur a constant cost per unit of time from queueing. In an equilibrium, a consumer joins a queue if its length is below a threshold level. The last consumer in the queue is just indifferent between joining and staying out. The equilibrium outcome is inefficient, due to the negative externality that a customer imposes on those arriving later. Nevertheless, rent dissipation through queueing is not full due to the randomness in the arrival and departure processes.

Richard Arnott et al. (1993) compare alternative toll regimes in a model with a single traffic bottleneck and identical commuters incurring linear time inconvenience cost (the "schedule delay cost"). Recent papers by Anderson and de Palma (2004) and Arnott and John Rowse (1999) study pricing of parking. Richard Porter (1977), writes on the optimal size of underpriced facilities, and points out that congestion is better than queuing, but does not seem to note the discontinuity, or actually solve for the optimal size.

The strategic incentives of people to adjust their purchase schedules are analyzed by

Robert Deacon and Jon Sonstelie (1991). Consumers choose the size of purchases to minimize the total cost of shopping for an underpriced good, which includes shopping and storage costs. The waiting time in a queue increases until the market clears. The authors note that consumers are no better off from a price ceiling, though suppliers are worse off, thus generating a deadweight loss in rationing by waiting. (See also Deacon & Sonstelie [1985, 1989] and Deacon [1994].)

Other economists have looked at similar problems; notably, William Vickrey, more famous for his work on auction theory. In "pure bottleneck" congestion in transportation facilities, queues form at a single route segment of fixed capacity. In Vickrey's 1969 pure bottleneck model, commuters have preferred travel times through the bottleneck, distributed uniformly. For a sufficiently large capacity, no queue develops and the commuters arrive at their preferred times. For a smaller capacity, this is not possible, a queue develops and some commuters arrive early or late. Each commuter faces a trade-off between the disutility of arriving at a less-preferred time and the cost of waiting in line. Vickrey notes a "sharp discontinuity" in the amount of delay at the level of capacity just sufficient to accommodate the traffic, and points out that optimal investments in capacity extension differ in the first-best and the second-best situations. In the first-best, using the optimal price structure (a toll fee that leads to efficient facility use), the benefits of capacity extension are not as "capricious" as in the second-best, when access cannot be restricted by fees. In the second-best, "Expansion inadequate to take care of the entire traffic demand...may turn out to be hardly worthwhile..." In Vickrey's model, however, capacity extension reduces delays and is beneficial to travellers. We will show that an insufficient increase in capacity might not have any benefit whatsoever to offset construction costs, so that the facility is a pure waste.

The idea closest to that modelled in the present paper is Steven Landsburg aquarium story in *The Armchair Economist*, in which all gains from a facility with free access are dissipated by congestion. Building a new aquarium does not benefit anyone while it is costly to build. In a way, the result is similar to the zero-profit outcome in a contestable market: potential competition (here by consumers) puts pressure on the active competitors. At the margin, the players' surplus is zero, and a given player is indifferent between using and not using the facility.

In the theory of waiting lists proposed by Lindsay and Feigenbaum (1984), market clearing occurs due to depreciation of product value in time. Since delivery of a service in future can be less valuable than immediate delivery, potential consumers are discouraged from putting their names on the list when it is too long. By this argument the authors explain the persistence of long waiting lists for non-emergency in-patient care at Britain's National Health Service and the fruitlessness of short-term measures aimed at a substantial reduction of the waiting list. See, however, the critique by Cullis and Jones (1986).

In our model, there are no waiting lists, congestion, or waiting in line. Rather, rent dissipation comes in the form of costly schedule adjustment by the travellers. In an effort to secure a parking spot, they come well in advance of their preferred time and dissipate nearly all rents whenever the number of drivers exceeds the number of parking spaces. The parking game under unobservability is closely related to all-pay auction models such as those of Holt and Sherman (1982), Barut and Kovenock (1998), and Clark and Riis (1998). These papers establish rent dissipation results for multi-prize all-pay auctions in which players make one-time decisions about their bids chosen from a continuous space.[2]

In the parking model, bids correspond to the cost of earlier arrival. When the game is played in time, players can possibly react to earlier moves by other players. Hence, we also examine the full observability case. Under observability, the mathematical structure of the parking game is related to preemption games and games of timing such as those of Fudenberg and Tirole (1985) and Simon and Stinchcombe (1989).

Our results will call to mind "jumping the gun" matching games such as those studied by Roth & Xing (1994) and Avery et al. (2001). To take the example of Avery et al., federal judges must each select one clerk from graduating law students, and students can work for no more than one judge. What has happened in recent years is that clerks and judges pair up earlier and earlier, rather towards the end of the student's last year of law school. A judge who waits too long to hire would not be able to find any good clerks still free, so

[2]In Holt & Sherman (1982) and Clark & Riis (1998), prizes are equal, but the papers differ in that Holt and Sherman assume incomplete information about players' valuations while Clark and Riis assume the information about asymmetric valuations to be complete. Barut and Kovenock (1998) consider symmetric valuations and an arbitrary prize structure. Clark and Riis (1998) also study sequential distribution of prizes, with a number of prizes distributed in each of several rounds. For a general analysis of equilibria in single-prize all-pay actions with complete information and asymmetric values, see Baye et al. (1996).

judges hire clerks early even though much better information is available about an older student's quality. The top students are analogous to the parking spaces in our model, and the possibility of mistakenly hiring an incompetent student is the cost of arriving early.

To sum up, we will describe competition for parking spaces as an all-pay auction because anyone who shows up early to look for a parking space has borne a cost even if he is unsuccessful. The auction is a multi-prize auction because there is more than one parking space. Since bids are reducing over time, the auction is a descending-price (Dutch) auction. Finally, the auction might be either open-cry or sealed-bid, depending on whether drivers observe other drivers' choices or not.

3. The Parking Game

In this section, we introduce the notation, set up the model, and discuss equilibria under various conditions.

3.1. The Model

A set of players – the drivers – is indexed by $I \equiv \{1, ..., N\}$. The drivers are workers who must show up for work no later than time, $t = T$. Each driver demands one space in the parking lot, and his value for it is $v > 0$, e.g., if he cannot find a spot, he must walk from somewhere further away at cost v. Let $K > 0$ be the size of the parking lot. If $N \leq K$, then each driver is guaranteed a parking space, and all drivers choose to arrive at their preferred time, T. If $N > K$, however, the drivers compete for spaces, and they may wish to arrive early to secure a spot.[3] Assume that a driver who arrives earlier than time T incurs a cost of $w > 0$ per unit of time, so his cost of arrival at time t is $L(t) = (T - t)w$. We will model time as discrete, with the interesting case being as the time interval $\Delta > 0$ shrinks to zero. Thus, a time grid includes times $t = k\Delta \in [0, T]$ where $k \in \{0, ..., T/\Delta\}$.

What matters to player i when deciding whether to rush to the parking lot at time t is the number of parking spaces unoccupied at t and the inconvenience of arrival as early as t. The history at time t can be summarized by the number of parking spaces, K_t, available.

[3] Parking competition has a form of first-in/first-out queuing. Other kinds of queues are also possible, if unobserved in the world. First-in/last-out queues can even lead to social optimality. See Hassin (1985) and its discussion by Nalebuff (1989).

We will assume that the size, N, of the parking lot is common knowledge, but it will be interesting to compare two alternative assumptions on whether a driver knows the number of spaces still open in the parking lot when he makes his own decision on when to arrive. We will denote the situation when a player making his arrival decision at time t observes all arrivals prior to t as *full observability*, and the situation when he must decide without knowing if the parking lot is full yet as *unobservability*.

For any time t, define N_t as the number of unarrived drivers and K_t as the number of parking spaces still open, so $N_t = N - K + K_t$. At time t, the remaining drivers simultaneously decide whether to arrive at the parking lot at that instant. Under full observability, the decision can be conditioned on the number of parking spaces, still available, K_t. Let $\sigma_{i,t} \in [0,1]$ denote the probability that player i arrives at the parking lot at time t. Once a parking spot is taken by the player, the spot remains occupied until the end of the game. If Player i arrives at t he obtains a spot with probability $p_{i,t} = \min\{K_t/n_t, 1\}$, where $n_t \leq N_t$ denotes the number of players arriving at exactly time t.

If Player i arrives at time t, given that $K_t \in \{0, ..., K\}$ parking spaces are unoccupied by t and $n_t \in \{0, ..., N_t\}$ players arrive at t, his payoff is

$$u_{i,t} = p_{i,t}v - L(t) \tag{1}$$

A player who arrives at time T and finds no parking space must park in a less convenient parking lot, obtaining payoff zero. This is the same payoff structure as for an all-pay auction with multiple prizes. Everyone bids, a bidder pays what he bids, and the prizes are distributed to the top bidders, with a coin toss breaking ties.

Let us define the *indifference arrival time*, t^*, as the time at which the arrival cost, $L(t^*)$, equals the prize value, v. It follows that

$$t^* \equiv T - v/w. \tag{2}$$

A player who arrives at t^* and finds a parking space receives a payoff of zero since the disutility of the early arrival equals the value of the parking space. Similarly, define t_p^* as the time at which a player receives a zero payoff from participating in a lottery with probability p of winning a parking space; t_p^* is found from equation $pv - L(t_p^*) = 0$. It follows that

$$t_p^* \equiv T - pv/w. \tag{3}$$

When looking for equilibria, we can restrict our attention to $t \geq t^*$ since arriving before t^* yields a negative payoff and is strictly dominated by arriving at T. Finally, let $\underline{t}$ and $\overline{t}$ denote the earliest and the latest time any player arrives with a positive probability along the equilibrium path.

As we are interested in characterizing equilibria which exist for any time grid, even a very fine one, we will assume that time periods are short.

Definition 1. *A time grid is called fine if*

$$\Delta < \frac{v}{(K+1)\,w}. \tag{4}$$

When a time grid is fine, there are more than $K+1$ periods between t^* and T since $T - t^* = v/w > (K+1)\Delta$. For a fine grid, a player would choose to arrive one period earlier to increase his odds of getting a parking space. The highest odds of obtaining a space for integer K_t and N_t are $K/(K+1)$. Even for these odds, the loss of the one-period earlier arrival, $w\Delta$, is justified by the gain associated with the higher probability of obtaining parking, $(1 - K/(K+1))v$. Also note that no more than K players can profitably enter at time $t = t^* + \Delta$ when the time grid is fine.

3.2. Pure-Strategy Equilibria Under Full Observability

Under full observability, Player i's strategy specifies the probability of his arrival, $\sigma_{i,t} \in [0,1]$ at any time $t = k\Delta \in [t^*, T]$ and history $K_t \in \{0, ..., K\}$. In the first best, all N players arrive at T, and K of them get to park in the parking lot: $\sigma_{i,t} = 0$ for $t < T$ and $\sigma_{i,T} = 1$. However, this cannot be a part of an equilibrium strategy profile when the time grid is fine since each player would prefer to deviate by arriving just before T and increase his probability of finding a spot to 100%. We will see that although the details of the equilibrium strategies can be various and intricate, especially for out-of-equilibrium behavior, the equilibrium payoffs converge to zero as time becomes closer to continuous: the parking lot will have no benefit to the drivers.

We will start by illustrating the logic of what happens in an *alternating equilibrium* in the simplest case of two drivers who compete for one parking space. The pure-strategy arrival schedule of Claim 1 specifies contingencies under which some players arrive, provided they have not arrived earlier; under all other contingencies, players do not arrive.

Claim 1. *The following are equilibrium strategies under full observability when there are two drivers and one parking spot ($N = 2$ and $K = 1$). Contingent on the parking lot not being full at time t, (i) at $t = t^* + k\Delta \in [t^*, t^*_{1/2})$: player 1 arrives if $k = 0, 2, 4, ...$ and player 2 arrives if $k = 1, 3, 5, ...$; (ii) at $t \in [t^*_{1/2}, T)$: both players arrive. At $t = T$: all players arrive who have not yet arrived.*

Proof of Claim 1. To prove that the listed strategies are part of a subgame-perfect Nash equilibrium we must show that there are no profitable deviations for any player at any point in time, given the strategy of the opponent.

Both players have equilibrium payoffs of zero. Time t^* is such that player 1 is indifferent between arriving at t^* and not parking in the lot at all. Arriving earlier than t^* (at $t < t^*$) yields the player a negative payoff, $v - L(t) < 0$. If player 1 deviates at t^* by delaying his arrival, player 2 arrives at $t^* + \Delta$. Arriving at $t^* + \Delta$, player 1 obtains $v/2 - L(t^* + \Delta) \leq 0$. Hence, player 1 cannot do any better than to arrive at T and receive a zero payoff.

A player assigned to arrive at $t \in (t^*, t^*_{1/2})$ will do so because otherwise the rival arrives next period. A one-period delay in player i's arrival would save the player the cost of arriving one period earlier, $w\Delta$, but will reduce player i's odds of obtaining parking from 1 to 1/2. For a fine grid, player i would prefer a secure space at t to a competition for the parking space at $t+\Delta$ with fifty-fifty odds. The parking lot is full in later periods. A player assigned to arrive at $t \in [t^*_{1/2}, T)$ will do so when the parking lot is not full at t because arrival at these times results in a nonnegative payoff even when the player's rival arrives at t for sure.

No player who is not assigned to arrive at $t \in [t^*, t^*_{1/2})$ would arrive at that time because this will yield a negative payoff to the player. By assumption, all players arrive no later than T. Q.E.D.

Player 1 arrives at t^*, Player 2 arrives at T, and both players obtain zero payoffs. The parking lot yields zero social benefit, and if it was costly to construct or the land has other uses, it has negative social value.

The equilibrium strategies also specify what happens out of equilibrium. In particular, suppose Player 1 fails to arrive at t^*. Player 2 then enters at $t^* + \Delta$ and obtains a positive payoff. Player 1 waits until T rather than trying to enter at $t^* + \Delta$, because he knows that Player 2 will be entering $t^* + \Delta$ and a fifty percent probability of getting a spot is not enough to justify showing up at $t^* + \Delta$. A player's arrival at any $t \in [t^*, t^*_{1/2})$ is sustained by the threat of arrival by the competitor at $t + \Delta$. For $t \in [t^*_{1/2}, T)$, arrival at t is weakly dominating any other time-t decision of a player.[4]

Let us now describe the pure-strategy subgame-perfect Nash equilibria for the game with N players. Denote by $\widehat{n}_t$ the critical number of players for whom entry yields a nonnegative payoff at time t, given that K_t spaces remain unclaimed by time t. The *free-entry* number of players, $\widehat{n}_t$, for given K_t and $t \geq t^*$ is determined as the largest n_t that satisfies condition $u_{i,t} = (K_t/n_t)\, v - L(t) \geq 0$. By definition, $u_{i,t^*} = v - L(t^*) = 0$, and therefore for a given K_t,

$$\widehat{n}_t \equiv \max \{n_t \in Z_+ : u_{i,t^*} \geq 0\} \tag{5}$$

Claim 2 describes all pure-strategy subgame-perfect equilibrium for the parking game on a fine time grid.

[4]The equilibrium strategies are not unique, although the equilibrium outcome is. There is another pure-strategy in which players alternate their out-of-equilibrium threats of arrival starting from time $t^* + \Delta$.

Claim 2.

A. *In any pure-strategy equilibrium under full observability and with more drivers than parking spaces, the parking lot is full after* $t^* + \Delta$ *when the time grid is fine and* $N > K$. *The equilibrium outcome is for* $K_0 \in \{0, ..., K\}$ *players arrive at* t^*; $K - K_0$ *players to arrive at* $t^* + \Delta$, and $N - K$ *players to arrive at* T.

B. *All the existing pure-strategy equilibria under full observability have the following arrival schedule for a fine time grid and more drivers than parking spaces:*

(i) at $t = t^*$, $K_0 \in \{0, ..., K_t\}$ *players arrive*

(ii) at $t = t^* + \Delta$, K_t *players arrive; if player* $i \in I$ *deviated by not arriving at* t^*, *the set of arriving players excludes player* i

(iii) at $t \in [t^* + 2\Delta, t^*_{1/2})$, *if player* $i \in I$ *deviated by not arriving at* $t - \Delta$, *one other player arrives*

(iv) at $t \in [t^*_{1/2}, T)$, *if player* $i \in I$ *deviated by not arriving at* $t - \Delta$, *then* $\min\{\widehat{n}_t, N_t\} \geq 2$ *players arrive at* t

(v) at $t = T$: *all players arrive who have not yet arrived.*

Proof of Claim 2. Consider a pure-strategy subgame-perfect Nash equilibrium. No player can arrive earlier than t^*. In part A, we need to show that the parking lot is full after $t^* + \Delta$ at the latest. Let t' denote the time when the parking lot becomes full and suppose that $t' \in [t^* + 2\Delta, T]$. If the odds of obtaining a parking space at t' are less then one for player i who is supposed to arrive at t', the player would instead arrive one period earlier. If a parking space is guaranteed at t', then a player who is supposed to arrive at T would instead arrive at $t' - \Delta$. Such players exist since $N > K$ and the number of arrivals prior to T is equal to the number of parking spaces. Hence, $t' \leq t^* + \Delta$.

The proof of part B is similar to that of Claim 1. Arriving earlier than t^* never benefits a player. If player i, deviates from his equilibrium strategy by not arriving at t^*, player $j \neq i$ arrives at $t^* + \Delta$, and player i receives a payoff of zero at best. For $t \in (t^*, t^*_{1/2})$, it is always better for a player to obtain a parking spot at t then to contest one remaining parking space next period or wait till later periods. At $t \in [t^*_{1/2}, T)$, a delay by the player assigned to arrive at t implies that the player does not obtain parking as the parking lot is filled at time t. The condition on the finesse of the time grid ensures that in case $K_0 = 1$, the player assigned to arrive at t^* deviates by arriving at $t^* + \Delta$, there are sufficient number of other players to take all the parking spaces at $t^* + \Delta$ and make the deviation unattractive to the player. Q.E.D.

Claim 2 shows that there are multiple equilibria under full observability, even if we restrict our attention to pure-strategy equilibria. Some players might arrive at the indifference arrival time t^*. This yields them zero payoffs and has the same effect on the remaining players as if the parking lot size K had shrunk and the game were started at $t^* + \Delta$. There are many sizes of this initial shrinkage that support equilibria. For a fine time grid, the equilibrium outcome in any pure-strategy equilibrium is for $K_0 \in \{0, K\}$ parking spaces to be filled at t^* and the rest to be filled at $t^* + \Delta$. There is a pure-strategy equilibrium in which one group of K players arrive at time t^* and they park in the lot, while a second group of $N - K$ players arrive at T and park elsewhere. Both groups receive the same payoff of zero. Another polar pure-strategy equilibrium is for no drivers to arrive at t^* and for K players to arrive at $t^* + \Delta$ as a pure strategy. In all pure-strategy equilibria, the parking lot fills up no later than $t^* + \Delta$ and no players arrive between $t^* + \Delta$ and T. An important corollary from Claim 2 concerns the extent of rent dissipation in the parking game under observability.

Corollary. *In the limit, if there are more drivers than parking spaces then as* $\Delta \to 0$ *they dissipate all the rents from parking.*

As the time grid becomes infinitely fine, players dissipate the entire value from the parking lot. In the limit, as $\Delta \to 0$, each driver has an expected payoff of zero and the total sum of all the losses incurred by drivers is equal to the total value of parking, vK. The parking lot must still be built, however, so the social payoff is negative and equal to the cost of building the parking lot.

There also exist mixed-strategy equilibria. These are complicated and varied enough that we will not fully describe the equilibria (including out-of-equilibrium behavior). However, in Section 4 we discuss the extent of rent-dissipation in any equilibrium to the parking game.

3.3. Non-Existence of Pure-Strategy Equilibria Under Unobservability

Under unobservability, Claim 2 does not apply. Claim 2 said that under full observability some players arrive at $t \leq t^* + \Delta$ and others arrive at T, both having nearly zero expected payoffs. Under unobservability, however, one of the players who is supposed to arrive at $t \leq t^* + \Delta$ could deviate and arrive at $T - \Delta$ instead, reducing his early-arrival cost – the other players would not observe that he had failed to arrive as scheduled at t^*, so they would be unable to respond by taking "his" spot before $T - \Delta$.[5]

Claim 3. *There does not exist a pure-strategy Nash equilibrium under unobservability when the time grid is fine and there are more drivers than parking spaces.*

Proof of Claim 3. Denote by t' the time when the parking lot becomes full in a pure-strategy equilibrium. If at t' not all arriving drivers obtain parking, then each of them would find it profitable to deviate by arriving a period earlier. If parking is guaranteed at t', then there are drivers who arrive at T because arriving between t' and T yields a negative payoff. Each of these drivers benefits from arriving shortly before t' unless $t' = t^* + \Delta$. A fine time grid implies that only K drivers would arrive at $t' = t^* + \Delta$. Then, each of them can arrive later and still obtain the parking space. Hence, no pure-strategy equilibrium exists in this case as well. Q.E.D.

The reason for the nonexistence of a pure-strategy Nash equilibrium under unobservability when the time grid is fine is the same as in an multi-prize all-pay auction with continuous strategy space (Barut & Kovenock, 1998). Players who do not obtain parking choose to arrive at their preferred time, and therefore other players do not have incentives to arrive much earlier. This, however, cannot be an equilibrium because players who do not always obtain parking prefer to bid enough to guarantee themselves a prize. Although mixed-strategy equilibria to the parking game under unobservability are difficult to fully describe,[6] in the

[5] From equilibria in Claim 2, one equilibrium survives but only when the time grid is *not* fine. The following is an equilibrium for the parking game under unobservability for $N > K$ and the time grid that is not fine, (i) At $t = t^* + \Delta$, $\min(\widehat{n}_t, N_t) > K$ players arrive; (ii) At T, all players arrive who have not yet done so. If any player deviated to arrive at t^*, he would not improve on his equilibrium payoff. If any player deviated to arrive after $t^* + \Delta$ but before T, he would not find a parking spot and would receive a negative payoff. This establishes that the specified strategies form an equilibrium.

[6] For instance, there is an alternating equilibrium in which for $\tau \in \{0, 1, 2...\}$ and $t \leq T$ player 1 arrives at $t = t_0 + 2\tau\Delta$ with probability $(v/(2w\Delta) - \tau)^{-1}$ and player 2 arrives at $t = t_0 + (2\tau + 1)\Delta$ with the same

next section we will show that almost complete rent dissipation occurs in any equilibrium of the parking game.

4. Full Rent Dissipation

We will first establish that in any equilibrium, players arriving at $\underline{t}$ and $\overline{t}$ (the earliest and the latest time any player arrives with a positive probability along the equilibrium path) obtain nearly zero payoffs whenever there are more drivers than parking spaces. Then, we will show that all players must almost fully dissipate the value from parking.

Claim 4. *In any equilibrium with more drivers than parking spaces, the probability that the parking lot is full at time $\overline{t} \leq T$ is one under full observability and approaches one as the time grid becomes infinitely fine under unobservability. The equilibrium payoffs of players who arrive at $\overline{t}$ with a positive probability are zero under full observability and they tend to zero under unobservability as the time grid becomes infinitely fine. Any player who arrives at $\underline{t}$ tends to dissipate all the rent from parking.*

The proof is in the Appendix. Claim 4 says that the parking lot is full almost surely at $\overline{t}$, and the payoffs of players who arrive at $\overline{t}$ and $\underline{t}$ are almost zero. It is nearly impossible to find parking at $\overline{t}$ because otherwise a player arriving at $\overline{t}$ could deviate by arriving at $\overline{t} - \Delta$ (while following the same strategy before $\overline{t} - \Delta$) and increase his odds of obtaining parking. An incentive to deviate exists whenever there is a parking space available at $\overline{t} - \Delta$ (under full observability) or when there is a nontrivial probability that parking is available at $\overline{t} - \Delta$ (under unobservability). Since there is almost no chance that a parking spot is available at $\overline{t}$, players' payoffs at $\overline{t}$ are nearly zero. Since $\overline{t} \leq T$, it follows that with a probability approaching one no parking spaces are available at the start of period T.

Corollary. *The parking lot is full by period T almost for sure.*

We next show that players dissipate all the rents from parking in any equilibrium to the parking game when the time grid is infinitely fine.

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probability. In case $w = 1$, $v = 10$, $T = 10$, and $\Delta = 1$, the probability of player 1 arriving at $t = 1, 3, 5, 7, 9$ and the probability of player 2 arriving at $t = 2, 4, 6, 8, 10$ is 1/5, 1/4, 1/3, 1/2, and 1. After $t = T - \Delta = 9$ the parking spot is full with probability one.

Proposition 1. *Under either full observability or unobservability, as the time grid becomes infinitely fine and there are more drivers than parking spots ($N > K$), players fully dissipate rents from the parking lot in any equilibrium.*

Proof of Proposition 1. Consider a subgame-perfect Nash equilibrium to the parking game under full observability or unobservability with $N > K \geq 1$. The proof is by contradiction. Suppose player i were earning a positive payoff bounded away from zero. Let $t \in (\underline{t}, \bar{t})$ be the earliest moment the player arrives with a positive probability; $t > \underline{t}$. Consider a player (player j) who arrives at $\bar{t}$ (with pure strategy) and earns an almost zero payoff. If player j follows its equilibrium strategy till $t - \Delta$ and arrives with probability one at $t - \Delta$, the player would obtain a positive payoff bounded away from zero. The costs of arriving at t and $t - \Delta$ differ by Δw, and the difference becomes negligible as $\Delta \to 0$. It suffices to prove that player j's odds of obtaining parking at $t - \Delta$ are no worse than player i's odds of obtaining parking at t. Note that by the definition of time t player i has not attempted to arrive before t. Player j may have been mixing before t. Therefore, player j has the expected payoff from arriving at $t - \Delta$ that is no less than that of player i. Since there is a profitable deviation, there cannot exist an equilibrium with a player obtaining a positive payoff at a time t between $\underline{t}$ and $\bar{t}$. Hence, all players almost dissipate the rents from parking. Q.E.D.

Proposition 1 shows that any player tends to fully dissipate rents from parking. The (nearly) zero payoffs present in all the equilibria suggest that the payoffs may be the most interesting feature of the parking game, and it is that feature that will be used when deciding on the optimal size of the parking lot.

5. Welfare and Parking Lot Size

Sections 3 and 4 described and analyzed the rent-seeking competition between drivers for a fixed number of parking spaces. We next characterize welfare more fully by adding consideration of the cost of building the parking lot. The drivers have a value $v > 0$ for each of the parking spaces in use. Let us denote the cost of providing K parking spaces by $C(K)$. Unless stated otherwise, we will assume that the production technology has a constant marginal cost, $C(K) = cK$. The welfare from a parking lot of size K is

$$W(K) = \begin{cases} vN - C(K) - E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right) & \text{if} \quad K \geq N \\ vK - C(K) - E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right) & \text{if} \quad K < N \end{cases} \tag{6}$$

where $E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right)$ denotes the expected cost to drivers of earlier than preferred arrivals in an equilibrium to the parking game for given K and N. Let us decompose welfare into two parts: the *provision cost* $C(K)$ and the *flow value*, which is either $vN - E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right)$ or $vK - E_{\sigma(N,K)}\left(\sum_{i=1}^{N} L(t_i)\right)$, depending on whether $K > N$ or not. The next two subsections deal with the problem of finding the optimal parking lot size in the cases of a certain and uncertain number of drivers for an infinitely fine time grid.

5.1. Optimal Capacity Under Certainty

The analysis of Section 4 shows that in competition for a limited number of parking spaces, drivers dissipate all the rents in the limiting case of an infinitely fine grid. In any equilibrium, they are just indifferent between arriving early enough to secure a spot and not parking in the parking lot at all. The expected cost of earlier arrivals equals zero when $K \geq N$ and it is vK when $K < N$. Hence, welfare is

$$W(K) = \begin{cases} vN - C(K) & \text{if} \quad K \geq N \\ -C(K) & \text{if} \quad K < N \end{cases} \tag{7}$$

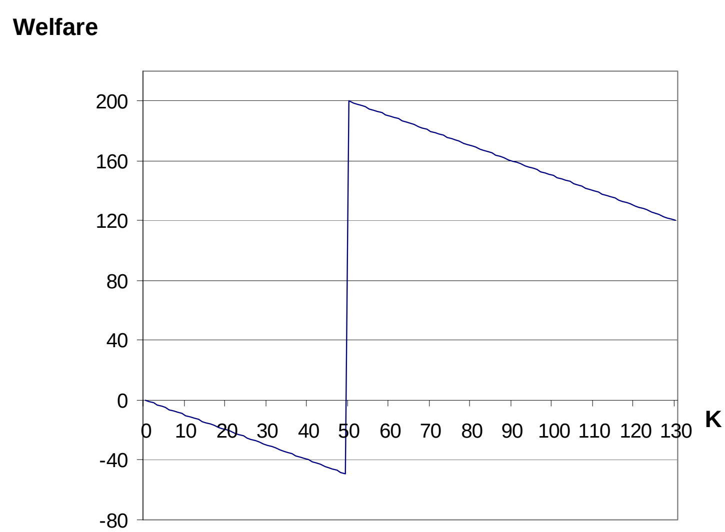
Full rent dissipation occurs each time the number of players exceeds the size of the lot. Therefore, to avoid wasteful rent-seeking activity, the parking lots have to be designed to accommodate all the people who need the parking. When the number of such people, N, is known with certainty, the parking lot should have N parking spaces. This result is summarized in Proposition 2 and illustrated by Figure 1.

Proposition 2. *Consider the parking game on an infinitely fine time grid when $v > c$. The optimal size of the parking lot under certainty equals the number of users, $K^* = N$. All smaller sizes have negative welfare, with the minimum welfare being at $K = N - 1$. All greater sizes have flow values equal to that for $K^* = N$, but increasingly high provision costs. This is true under both full observability and unobservability.*

Proof of Proposition 2. If $K < N$, then in any equilibrium to the parking game almost all rents are dissipated and $W(K) = -C(K)$. If $K \geq N$, each player is guaranteed a parking space and arrives at the preferred time T. Therefore, $W(K|K \geq N) = vN - C(K)$. It follows that welfare is maximized at $K = N$ as long as it is socially beneficial to build a parking lot at all, i.e. when $v > c$. Q.E.D.

According to Proposition 2, the size of the parking lot should be set at $K = N$, as long as it is socially beneficial to build a parking lot at all. This result becomes clear when looking at the shape of the welfare function in Figure 1. There is a sharp discontinuity in social welfare between "enough" and "not enough," with the minimum social welfare being at slightly too small a parking lot.

Figure 1. The Welfare from a Parking Lot of Size K when $N = 50$, $c = 1$, and $v = 5$



The next subsection deals with the case of uncertain number of drivers. Interestingly, it will show that uncertainty over the number of drivers actually increases social welfare if the parking lot size is set too small.

5.2. Optimal Capacity under Uncertainty

Assume that all drivers know the demand for parking before they choose their arrival times, but the planner has to decide on the size of the parking lot before uncertainty about demand (the value of N) is resolved. Maximization of expected welfare then requires a much bigger parking lot than the expected value of potential demand, because the loss function in Figure 1 is asymmetric. In a stochastic model, there will be usually many empty spaces in a lot of optimal size. Planners should not be tempted to increase the number of parking permits, although unsophisticated observers may decry the wastefulness of building parking lots that are too large or of cruelly limiting the number of permits despite the presence of unused parking spaces.

Suppose the number of players who seek parking is uncertain and is drawn from a known probability distribution, $f(N)$. What is the optimal size of the parking lot? When there is competition for parking spots, $K < N$, the benefit from the parking lot is negative, $W = -cK$. When the size of the lot is large enough, $K \geq N$, the benefit to N players is $W = vN - cK$. Expected welfare is

$$EW(K) = v\sum_{N=0}^{K} Nf(N) - cK = vE(N \mid N \leq K) - cK \tag{8}$$

The size of the parking lot should be increased as long as the marginal net benefit is non-negative

$$EW(K) - EW(K-1) = vKf(K) - c \geq 0 \tag{9}$$

Since drivers benefit from the Kth parking space only if $N = K$, the welfare increase equals the probability that there are exactly K drivers multiplied by the extra benefit from eliminating rent-seeking behavior, vK, net of the cost. The intuition is that it is more important to have a big parking lot as K gets bigger, because there are more people who could get benefit from it. At the same time, it could be less likely that larger parking lots are filled out. Whether the marginal benefit of a parking space is decreasing or increasing depends on the relative strengths of the two effects.

Inequality (9) can be rewritten as

$$Kf(K) \geq c/v \tag{10}$$

Expansion of the parking lot is welfare-improving if the probability that exactly K drivers compete for K parking spaces times the size of the parking lot exceeds the relative cost of building a parking space, c/v. Next we apply the theory to the uniform and binomial distributions of N, and present the analysis of the parking problem when the number of drivers is large and calculus methods can be employed.

Example 1: The Uniform Distribution

Consider a discrete uniform distribution on the support $\{0, ..., \overline{N}\}$ with p.d.f. $f(N) = 1/(\overline{N}+1)$. Notice that for larger capacity levels, the benefit from building an additional space in equation (9) is higher. It is equally likely at any capacity that the parking demand will be just met, and at a larger capacity benefits accrue to more people. Hence, there is no interior solution. With a uniform distribution, the parking lot should be big enough to include all the people who might possibly want to park, if it should be built at all. A comparison between $EW(0) = 0$ and $EW(\overline{N}) = (v/2 - c)\overline{N}$ reveals that the parking lot of size large enough to accommodate all potential demanders should be constructed as long as $v/2 - c > 0$. On average, 50% of the parking spaces will be unclaimed since $E(N) = \overline{N}/2$. A parking lot is desirable if the expected value of a spot, $v/2$, exceeds its cost, c.[7]

Under uncertainty, the expected welfare is $EW(K|K < \overline{N}) = (v/2)(K+1)K/(\overline{N}+1) - cK$ and $EW(K|K \geq \overline{N}) = v\overline{N}/2 - cK$. We can compare the expected welfare at different capacity levels under uncertainty to the welfare under certainty (with the expected number of drivers, $\overline{N}/2$, arriving for sure). When $\overline{N}/2$ drivers are arriving to the parking lot with certainty, the welfare is $W(K|K < \overline{N}/2) = -cK$ and $W(K|K \geq \overline{N}/2) = v\overline{N}/2 - cK$. Table 1 and Figure 2 show the expected welfare for the uncertainty case and welfare for the certainty case at different capacity levels when $c = 1, v = 5$, and $\overline{N} = 100$.

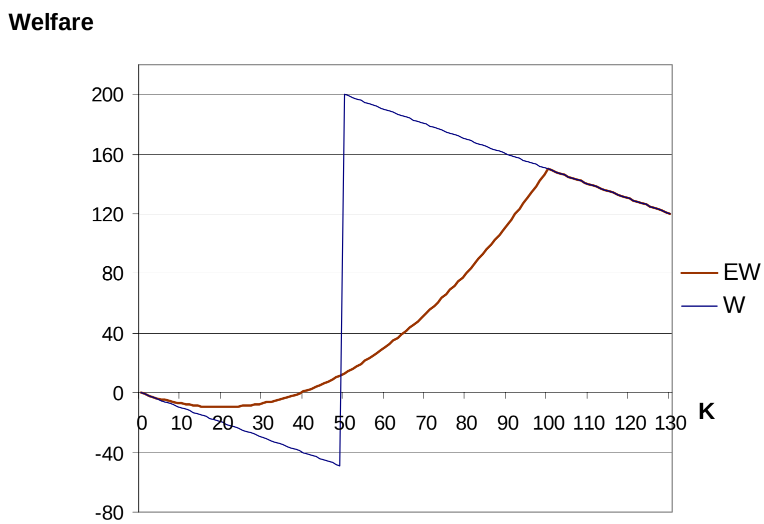
[7] For any probability distribution, $f(\cdot)$, such that the optimal capacity size is equal to the upper bound of the support of the distribution, capacity utilization is equal to the ratio of the expected number of drivers to the maximum number of drivers, $E(X) = E(N)/\overline{N}$. The parking lot should be build only if the expected value is no less than the cost, $vE(N) \geq c\overline{N}$.

Table 1. Welfare and Parking Lot Size when the Number of Drivers Is Uniformly Distributed

Number of Spaces, K	0	10	20	30	40	50	60	70	80	90	100
Expected Welfare (Uncertainty)	0	-7	-10	-7	1	13	31	53	80	113	150
Welfare (Certainty, $N = 50$)	0	-10	-20	-30	-40	200	190	180	170	160	150

Notes: Welfare is rounded to the nearest integer.

Figure 2. Welfare and Parking Lot Size when the Number of Drivers Is Uniformly Distributed



Compare the consequences of a limited capacity for certain and uncertain N. Uncertainty over the number of drivers actually increases social welfare if the parking lot size is set too small. While the optimal capacity under uncertainty is $K^* = 100$, even if $K = 40$, welfare is positive. This is a big difference from the case without uncertainty, where it would be near its minimum and very negative. The reason is that under uncertainty, even with very few parking spaces, it may happen that very few people need to park, and so there is no wasteful rent-seeking and the parking spaces are valuable.

This is somewhat paradoxical. In the first-best solution, when people are allocated to parking spots, welfare would be highest under certainty since no parking spot would ever go unused and no driver would ever fail to find a parking spot. The same is true when drivers are not strategically changing their arrival schedules in attempt to secure a spot. Uncertainty would make some excess capacity optimal, but would reduce welfare. If people are strategic, however, the consequences of mistaken policy lead to quite different outcomes. Having a slightly too small a parking lot would be disastrous under certainty, with zero flow payoff, but under uncertainty the flow payoffs would be positive. Importantly, empty parking spaces are not an indication that the parking lot is big enough. That will happen sometimes even in a very inefficient equilibrium.

Example 2: The Binomial Distribution

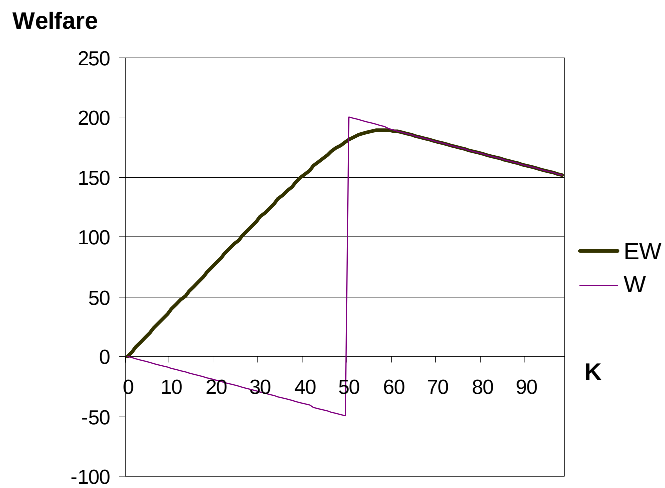
Consider a binomial distribution, which arises when drivers' needs for parking are independent random trials and suppose that each of 100 drivers will need parking with probability θ. A numerical example for $c = 1, v = 5, \overline{N} = 100$, and $\theta = 0.5$ is used as an illustration. For these parameter values, Table 2 and Figure 3 show the expected welfare for the uncertainty case and welfare for the certainty case at different capacity levels.

Table 2. Welfare and Parking Lot Size when the Number of Drivers Is Binomially Distributed

Number of Spaces, K	0	10	20	30	40	50	60	70	80	90	100
Expected Welfare (Uncertainty)	0	39	78	117	153	182	189	180	170	160	150
Welfare (Certainty, $N = 50$)	0	-10	-20	-30	-40	200	190	180	170	160	150

Notes: Welfare is rounded to the nearest integer.

Figure 3. Welfare and Parking Lot Size when the Number of Drivers Is Binomially Distributed



In this example, the size of the parking lot should be $K^* = 58$. Only about 8 out of 58 spaces are empty, on average. This corresponds to 86% utilization level.

Example 3: A Large Number of Drivers: The Calculus Approach

For large N we can abstract from the integer problem and use calculus. The expected welfare from the parking lot of size K is then

$$EW(K) = v \int_0^K N f(N) dN - cK \tag{11}$$

The optimal size of the parking lot is the solution to the first-order condition $\partial EW(K)/\partial K = vKf(K) - c = 0$, which can be re-written as

$$Kf(K) = c/v \tag{12}$$

We can use (12) to find the optimal level of K if the maximand in (11) is concave. Surprisingly, that seems unlikely. The second-order condition, $\partial^2 EW(K)/\partial K^2 < 0$, requires $Kf'(K) + f(K) < 0$. This means that the probability distribution function has to be declining faster than $1/K$ in the relevant range of K. This is true for some distribution functions, however, as we will see next.

Example. Consider a continuous p.d.f. $f(N) = N^{-\beta}$ with $\beta \in (1, 2)$, defined on the support $[a, b] = [1, (2-\beta)^{\frac{1}{1-\beta}}]$. Note that $\int_a^b N^{-\beta} dN = 1$ and the mean number of drivers in need of parking is $E(N) = \int_a^b N f(N) dN = \left(b^{2-\beta} - 1\right)/(2-\beta)$.

The first-order condition implies the optimal capacity $K^* = (c/v)^{1/(1-\beta)}$, and the second-order condition is satisfied. The mean utilization of the parking lot of optimal size can be measured as a ratio of the mean number of parking spots taken, $E(X)$, to the optimal capacity size, K^*. If $N < K$, then all N drivers find parking; if $N \geq K$, then K out of N drivers find parking. Hence, $E(X) = \int_a^{K^*} N f(N) dN + \int_{K^*}^b K^* f(N) dN$. The mean capacity utilization as a percentage of the optimal lot size is, therefore, $E(X)/K^* = \left((c/v) - (c/v)^{\frac{1}{\beta-1}} (\beta - 1) - (2-\beta)^2\right) / \left((2-\beta)(\beta-1)\right)$. For example, when $c/v = 0.5$ and $\beta = 1.5$, $K^* = 4$ and $E(X) = 2$. (Keep in mind that the number of drivers must be large for the analysis of this section to be correct, so "4" might denote 4 thousand drivers.) On average, 50% of the parking spots will remain unoccupied. The 50%-utilized parking lot is socially optimal.

The expected welfare for a parking lot of size K, relative to the value v of parking, is $EW(K)/v = \int_1^K N^{1-\beta} dN - (c/v)K = \left(K^{2-\beta} - 1\right)/(2-\beta) - (c/v)K$. To compare the results to those under no uncertainty, suppose that the distribution for N is degenerate, taking value $N = E(N)$ for sure. Under no uncertainty, the optimal lot size is $E(N)$. The welfare from the parking lot is $W(K = E(N)) = (v - c)E(N)$.

Table 3 compares welfare at different capacity levels for a variety of parameter combinations.

Table 3. Capacity Utilization and Expected Welfare

	Parameter Values								
Welfare Measures	$c/v = 0.4$			$c/v = 0.5$			$c/v = 0.6$		
	$\beta = 1.6$	1.7	1.8	$\beta = 1.6$	1.7	1.8	$\beta = 1.6$	1.7	1.8
$E(N)$	2.11	2.25	2.48	2.11	2.25	2.48	2.11	2.25	2.48
K^*	4.61	3.70	3.14	3.17	2.69	2.38	2.34	2.07	1.89
Capacity Utilization (%)	45.71	57.59	65.95	62.92	71.41	77.28	76.63	82.18	85.97
[a]Welfare, $W(E(N))/v$	1.26	1.35	1.49	1.05	1.13	1.24	0.84	0.90	0.99
[b]Welfare, $EW(K^*)/v$	2.11	1.60	1.29	1.47	1.15	0.95	1.01	0.82	0.68
[c]Welfare $EW(K'')/v$	0.26	0.12	0.03	- 0.12	-0.19	-0.24	-0.39	-0.43	-0.46

Notes: K^* is the optimal size of the parking lot under uncertainty. a) No uncertainty, $K = E(N)$; b) Uncertainty, $K = K^*$; c) Uncertainty, $K'' = 0.99K^*$ - capacity is 1% below optimal.

Table 3 illustrates our assertion that a slightly too small parking lot can be worse than no parking lot at all. In case (c), the size of the parking lot is one percent below the optimal level, and this small change greatly affects the expected welfare.

6. Discussion of Inter-Driver Contracting, the Assumption of Discrete Time, and the Parking Lot Cost Function

The rent-seeking behavior of drivers can be avoided by either increasing the capacity level or by restricting the entry to the facility. Under certainty, if there are K parking spaces and $N > K$ people willing to park, access should be restricted to K people. If more people were given parking permits, the competition would not allow them to obtain benefit from parking. The effect of extra parking permits would be to reduce the benefit for the original permit holders to zero.

If binding contracts could be made, the problem would be avoided. Everyone would arrive at T, K people would take the parking spaces, and the other $N-K$ would get side payments from the ones who park in the desirable lot. This, however, requires (a) communication to coordinate who parks, (b) low enough transaction costs for coordinating and making the payments, and (c) enforceability at low cost, so that people do not break their contract and arrive early or refuse to make the side payments later.[8]

We have modeled the parking game in discrete rather than continuous time for a number of reasons. A discrete-time framework is better describing situations when players can choose time of arrival from a given set of permissible times. For example, when the model is interpreted as describing the timing of purchases by bargain hunters, the decision may be to arrive on Friday, Saturday, or Sunday during the sales period. When time is continuous, there are no "alternating" equilibria of the kind described in Claims 1 and 2 since there does not exist "the next time." An attractive property of discrete-time equilibria is that they permit two or more players to arrive at the same time. The probability of such a tie is zero for atomless strategies in continuous time. Moreover, continuous time may be considered a limiting case of the discrete time as the time grid becomes infinitely fine.

One way to specify strategies in continuous time under unobservability would be to consider cumulative distribution functions for players' arrival times. This approach results in complete rent dissipation in any equilibrium, as Barut and Kovenock (1998) demonstrate. It follows from Theorem 2 in Barut and Kovenock (1998) that the expected payoff of any player is zero in any (mixed-strategy) equilibrium under unobservability when $N > K \geq 1$. Hence, under unobservability the full rent dissipation result extends to the continuous-time framework. Under full observability, players' strategies have to be conditioned on the history of arrivals to the parking lot, which cannot be captured with the traditional approach of finding equilibrium cumulative distribution functions for players' arrival times, and presents technical difficulties.

It is easy to extend the analysis of the optimal capacity choice to allow for an arbitrary cost function, $C(K)$. Due to the full rent dissipation result, the planner would still choose

[8] In 1999 a Ticket Master in Windsor, Ontario, used an interesting approach to distributing tickets. To prevent the practice of overnight lineups and discourage scalping a random number line-up procedure was adopted. This approach assigns a random number to each ticket buyer present at the time the office is opening and the queue is formed accordingly. While the procedure may appear unfair to many ticket buyers, it eliminates incentives to arrive early camping over night or long hours of waiting in line by providing each customer with a fair and equitable opportunity to be first in line. The case is reported, among with many other queueing examples, at http://www2.uwindsor.ca/˜hlynka/qreal.html.

to guarantee each driver a parking space when N is certain, or not build the parking lot at all.[9] In contrast, we conjecture that when drivers value parking spots asymmetrically, guaranteeing every driver a spot may not be optimal. The social planner has to choose the capacity, K, to maximize the welfare, measured by the cumulative value of the parking lot to drivers net of the cost of capacity. The planner faces a trade-off between the production cost and the cost of wasteful rent-seeking. When capacity is at least as high as the number of players, players are guaranteed parking spaces and there is no competition for the spaces. Since extra spaces are costly, no excess capacity is build in the case of the certain number of players. The planner chooses between guaranteeing a parking space for each driver and staging a contest, and even under certainty, a contest may be preferred if the cost is convex.[10]

7. Concluding Remarks

We have constructed various versions of a parking lot model, focusing not so much on planning for uncertain demand as on planning for strategic behavior by the demanders. Suppose 1,001 drivers want to arrive at the same time and have the same costs of arriving early, and each derives a benefit of \$250 from parking there during the year. Then if the cost of a parking space is \$200 per year, it is obvious that 1,001 spaces should be built, for a net payoff of \$50,050 per year. What is not so obvious, and what has been the theme of this paper in various models, is that if 1,000 spaces are built instead, the net payoff is not \$50,000, but rather -\$200,000. Competition in the form of early arrival for the scarce spots eats up the entire benefit from the parking lot. Of course, the implications of a shortage may

[9] When parking spaces are of different values, for example, due to variations in proximity and/or convenience, the rent dissipation result may still hold. For all-pay auctions cast in continuous space under unobservability, Barut and Kovenock (1998) establish full rent dissipation for arbitrary prize valuations, provided all players are not guaranteed the same prize value.

[10] Clark and Riis (1998) establish a rent dissipation result for a multi-prize all-pay contest with asymmetric valuations for the unobservability case and continuous strategy space. Their Proposition 1 implies that the expected net surplus from K identical parking spaces, valued as v_i by player i, is $\sum_{i=1}^{K}(v_i - v_{K+1})$ when $K < N$. Hence, in this framework the expected welfare is $W(K|K<N) = \sum_{i=1}^{K}(v_i - v_{K+1}) - C(K)$. The planner would increase the size of the parking lot from $(K-1)$ to K if the marginal benefit is no less than the marginal cost, i.e., if $C(K) - C(K-1) \leq (v(K) - v(K+1))\,K$. Staging the contest between drivers for parking spaces is preferred when $W(K|K<N) > W(K=N)$, which can be written as $C(N) - C(K) \geq \sum_{i=K^*+2}^{N} v_i + \sum_{i=1}^{K^*+1} v_{K^*+1}$. The first summation term corresponds to an increase in the number of beneficiaries from K^* to N and the second summation term is due to the lack of rent-seeking when parking is guaranteed.

not be as dramatic when, for example, the number of drivers is uncertain, but this extreme case shows the nature of the problem.

Thus, strategic incentives are an essential element in planning capacity for an underpriced good – as important, or perhaps more important, than the obvious decision-theory problem of predicting uncertain demand and the obvious engineering problem of predicting capacity cost. If for some reason direct pricing is impractical, and the planner is aware that some of the time demand for the good could exceed the supply, he should realize that the damage from such situations in not limited to just a few people finding the good has run out, because people's actions to forestall being thus shut out can vastly increase the damage. Civil engineers need to understand game theory.

Appendix: Proof of Claim 4

Claim 4. *In any equilibrium with more drivers than parking spaces, the probability that the parking lot is full at time $\bar{t} \leq T$ is one under full observability and approaches one as the time grid becomes infinitely fine under unobservability. The equilibrium payoffs of players who arrive at $\bar{t}$ with a positive probability are zero under full observability and they tend to zero under unobservability as the time grid becomes infinitely fine. Any player who arrives at $\underline{t}$ tends to dissipate all the rent from parking.*

Proof. Consider a subgame-perfect equilibrium to the parking game under full observability or unobservability. By the definition of $\bar{t}$, there is a player who does not arrive with a pure strategy before $\bar{t}$. The player (player i) who some times arrives at $\bar{t}$ should be unwilling to deviate by arriving at $\bar{t} - \Delta$ with a pure strategy, while keeping his arrival schedule before $\bar{t} - \Delta$ the same. Note that $\bar{t}$ is either equal to the time when the parking lot becomes full for sure, denoted by t', or the preferred time, T, since no players arrive between t' and T.

Player i's payoff from arriving at $\bar{t}$, for given $K_{\bar{t}-\Delta}$ and the number of other players who happen to arrive at $\bar{t} - \Delta$, n, is

$$u_{i,\bar{t}}|_{K_{\bar{t}-\Delta},n} = p_{i,\bar{t}} \cdot v \tag{A1}$$

where

$$p_{i,\bar{t}} = \frac{K_{\bar{t}}}{N_{\bar{t}}} < 1 \tag{A2}$$

is the probability that player i obtains parking at $\bar{t}$, $K_{\bar{t}} = \max\{K_{\bar{t}-\Delta} - n, 0\}$ and $N_{\bar{t}} = N - K + K_{\bar{t}}$.

For given $K_{\bar{t}-\Delta}$ and n, player i's payoff from arriving at $\bar{t} - \Delta$ is

$$u_{i,\bar{t}-\Delta}|_{K_{\bar{t}-\Delta},n} = p_{i,\bar{t}-\Delta} \cdot v - w\Delta \tag{A3}$$

where

$$p_{i,\bar{t}-\Delta} = \min\left\{\frac{K_{\bar{t}-\Delta}}{n+1}, 1\right\} \tag{A4}$$

is the probability that player i obtains parking at $\bar{t}-\Delta$ conditional on $K_{\bar{t}-\Delta}$ parking spaces available at $\bar{t}-\Delta$ and n other players arriving at $\bar{t}-\Delta$.

Three possibilities arise for different values of $K_{\bar{t}-\Delta}$ and n.

Case 1: If $K_{\bar{t}-\Delta}=0$, then $p_{i,\bar{t}}=0$ and $p_{i,\bar{t}-\Delta}=0$.

Case 2: If $0\leq n<K_{\bar{t}-\Delta}$, then $p_{i,\bar{t}}=K_{\bar{t}}/(N-K+K_{\bar{t}})<1$ and $p_{i,\bar{t}-\Delta}=1$.

Case 3: If $n\geq K_{\bar{t}-\Delta}>0$, then $p_{i,\bar{t}}=0$ and $p_{i,\bar{t}-\Delta}=K_{\bar{t}-\Delta}/(n+1)>0$.

Cases 1-3 show that unless $K_{\bar{t}-\Delta}=0$, arriving at $\bar{t}-\Delta$ provides player i with a strictly higher odds of obtaining parking. If $0\leq n<K_{\bar{t}-\Delta}$, then $p_{i,\bar{t}-\Delta}-p_{i,\bar{t}}=(N-K)/(N-K+K_{\bar{t}-\Delta}-n)>0$. If $n\geq K_{\bar{t}-\Delta}>0$, then $p_{i,\bar{t}-\Delta}-p_{i,\bar{t}}=K_{\bar{t}-\Delta}/(n+1)>0$. Finally, if $K_{\bar{t}-\Delta}=0$, then $p_{i,\bar{t}-\Delta}-p_{i,\bar{t}}=0$.

Under full observability, player i's deviation is not profitable for a given $K_{\bar{t}-\Delta}$ if inequality $u_{i,\bar{t}}|_{K_{\bar{t}-\Delta}}-u_{i,\bar{t}-\Delta}|_{K_{\bar{t}-\Delta}}\geq 0$ holds. The inequality can be written as

$$\sum_{n=0}^{N_{\bar{t}-\Delta}-1}\Pr(n)\cdot\left(p_{i,\bar{t}-\Delta}-p_{i,\bar{t}}\right)\cdot v\leq w\Delta. \tag{A5}$$

Inequality (A5) always holds if $K_{\bar{t}-\Delta}=0$ since player i does not increase his odds of obtaining parking when the parking lot is already full at $\bar{t}-\Delta$. When $K_{\bar{t}-\Delta}=0$, player i chooses to arrive at T. If there exists a parking space at $\bar{t}-\Delta$ (i.e., $K_{\bar{t}-\Delta}>0$), a player who is supposed to arrive at $\bar{t}$ will always deviate by arriving with a pure strategy at $\bar{t}-\Delta$, keeping the arrival schedule before $\bar{t}-\Delta$ unchanged. This is because the deviation increases player i's odds of obtaining parking by a positive amount bounded away from zero and the additional cost of the earlier arrival approaches zero. In other words, since the right-hand side of (A5) goes to zero as time grid becomes infinitely fine, the left-hand side of the inequality has to be converging to zero as well. This is not possible for $K_{\bar{t}-\Delta}>0$ since $p_{i,\bar{t}-\Delta}-p_{i,\bar{t}}\nrightarrow 0$ and $\Pr(n)\nrightarrow 0$ for some n. To summarize, if $K_{\bar{t}-\Delta}=0$, player i arrives at $\bar{t}-\Delta$ while if $K_{\bar{t}-\Delta}>0$, player i arrives at T. This is consistent with our assumption that player i arrives at $\bar{t}$ with a pure strategy (conditional on not arriving before $\bar{t}$) only if $\bar{t}=T$. Consider the case where $\bar{t}=T$. At least two players arrive at T with a pure strategy, conditional on not arriving before T. If $K_{T-\Delta}=0$ then $K_T=0$ as well, and players arriving at T receive a zero payoff. If $K_{T-\Delta}>0$, then all players who have not arrived choose to arrive at $T-\Delta$

since the odds of obtaining a parking spot at $T-\Delta$ are higher than at T in this case. Once again, we obtain $K_T = 0$. Hence, under full observability $K_{\bar{t}} = 0$.

Under unobservability, player i's deviation is not profitable if inequality $u_{i,\bar{t}} - u_{i,\bar{t}-\Delta} \geq 0$ holds, which can be written as

$$\sum_{K_{\bar{t}-\Delta}=0}^{K} \sum_{n=0}^{N_{\bar{t}-\Delta}-1} P_{i,\bar{t}-\Delta}(K_{\bar{t}-\Delta}) \Pr(n) \cdot \left(p_{i,\bar{t}-\Delta} - p_{i,\bar{t}}\right) \cdot v \leq w\Delta \tag{A6}$$

where $P_{i,\bar{t}-\Delta}(K_{\bar{t}-\Delta})$ denotes player i's assessment of the probability that there are $K_{\bar{t}-\Delta}$ empty spaces at $\bar{t}-\Delta$, given player i has not arrived before $\bar{t}-\Delta$. Since the right-hand side of the inequality goes to zero as time grid becomes infinitely fine, the left-hand side of the inequality has to be converging to zero as well.

Suppose that in an equilibrium $K_{\bar{t}-\Delta} > 0$ arises with a positive probability bounded away from zero (i.e., there exists $K_{\bar{t}-\Delta} > 0$ such that $P_{i,\bar{t}-\Delta}(K_{\bar{t}-\Delta}) \nrightarrow 0$). Since $p_{i,\bar{t}-\Delta} - p_{i,\bar{t}} > 0$ for $K_{\bar{t}-\Delta} > 0$, we find that $\Pr(n) \to 0$ for all possible realizations of n in the equilibrium for given $K_{\bar{t}-\Delta}$. This is not possible since $\sum_{n=0}^{N_{\bar{t}-\Delta}-1} \Pr(n) = 1$. Hence $P_{i,\bar{t}-\Delta}(K_{\bar{t}-\Delta}) \to 0$ for all $K_{\bar{t}-\Delta} > 0$. Under unobservability we find that almost for sure no parking is available to player i arriving at $\bar{t}$.

Suppose a player arriving at $\underline{t}$ with a positive probability were to obtain a positive payoff bounded away from zero. This implies that $\underline{t} > t^* + \Delta$. For $\underline{t} > t^* + \Delta$, the player with a nearly zero payoff at $\bar{t}$ can increase his payoff by arriving at $\underline{t} - \Delta$. Hence, the payoff of the player arriving at $\underline{t}$ approaches zero.[11] Q.E.D.

[11] It is easy to show that the earliest and the latest time anyone arrives with a positive probability in an equilibrium approach t^* and T respectively as the time grid becomes infinitely fine. First, we show that $\underline{t} \to t^*$ as $\Delta \to 0$. If $\underline{t} \leq t^* + \Delta$, then $\underline{t} \to t^*$. Suppose that $\underline{t} > t^* + \Delta$. At $\underline{t}$, all parking spaces are available, and the probability of obtaining a space at $\underline{t}$ converges to one because otherwise a player assigned to arrive at $\underline{t}$ would arrive at $\underline{t} - \Delta$. Since the payoff at $\underline{t}$ approaches zero, $\underline{t} \to t^*$ in this case as well. Second, we show that $\bar{t} \to T$ as $\Delta \to 0$. Since it is nearly impossible to obtain parking at $\bar{t}$ in an equilibrium and players receive non-negative payoffs, the cost of arriving at $\bar{t}$ must be zero in the limit as well. Hence, $\bar{t} \to T$ as $\Delta \to 0$.

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