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Strategic Complementarities and Search Market Equilibrium[†]

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In this paper, we apply supermodular game theory to the equilibrium search literature with sequential search. We identify necessary and sufficient conditions for strategic complementarities and prove existence of search market equilibrium. When firms are identical, the Diamond Paradox obtains and is robust within the class of search cost densities that are small near zero and support strategic complementarities. Price dispersion is therefore inherently incompatible with strategic complementarities. Finally, we show that a major criticism of the literature, that agents act as if they know the distribution of prices, can be justified in the sense of convergent best response dynamics.

Keywords: Diamond Paradox, price dispersion, search, strategic complementarities.

JEL Classification Numbers: D43, D83.

[†] I thank participants at the 2006 Econometric Society Meetings in Vienna and seminars at Indiana and Purdue for their comments. The usual disclaimer applies.

1. Introduction

In this paper, we apply concepts and tools from the literature on supermodular games to address three fundamental long-standing open questions in the equilibrium search literature with sequential search. The first is: what is the nature and source of observed price dispersion, which has been extensively documented in real-world markets? According to conventional wisdom, distilled from Rob's (1985) seminal contribution, price dispersion may be a purely informational phenomenon in the sense that it can arise solely from imperfect information about market prices and heterogeneity in consumers' search costs under mild assumptions.

In this paper, we consider the general class of equilibrium search models with sequential search developed by Carlson and McAfee (1983) and Bénabou (1993), with potential bilateral heterogeneities in consumers' search costs as well as firms' production costs. An important distinction between our model and that in Rob (1985) is that the latter takes demand to be perfectly inelastic with infinite willingness to pay, which rules out *ex ante* the Diamond (1971) equilibrium where almost all firms charge the unique monopoly price. In our model, we show that when firms are identical and a Rob-type condition applies to the density of search costs at zero then the Diamond equilibrium is the unique search market equilibrium and the Diamond Paradox is robust to heterogeneity in search costs within the class of search cost distributions that generate strategic complementarities. Although heterogeneous search costs may contribute to observed dispersion, they are not the source of it, and dispersion is not a purely informational phenomenon under these conditions. We then show that a simple Reinganum (1979)-type condition on production costs *is* sufficient for all search market equilibria to exhibit dispersion. In that case, dispersion is driven at least in part by technological conditions.¹

Since its inception in the early 1960s, a major recurring criticism of the literature has

¹ The above discussion is limited to equilibrium search models with sequential search. In models with other search technologies, dispersion can obtain in equilibrium even though consumers and firms are *ex ante* identical. See Baye, Morgan, and Scholten (2006) for a survey of the theoretical literature and empirical evidence on price dispersion.

been that consumers and firms act “as if” they know the distribution of prices. E.g.,

These results depend on the assumption that the searcher behaves as if he knows the distribution of prices. In any economic context, this is a very bad assumption. Little is known about the nature of price distributions, and it seems absurd to suppose that consumers know them with any degree of accuracy.

Rothschild (1974)

and

Perhaps the most restrictive and least palatable assumption of the elementary search model is the supposition that the offer distribution F is known.

Lippman and McCall (1981).

Although this criticism is valid for any static game-theoretic equilibrium concept, it applies with particular force to this literature, whose explicit aim after all is to characterize markets with informational imperfections. Moreover, it seems clear that real-world consumers (and firms) do not have anything close to the required information.

To address this issue, Bénabou and Gertner (1993) and Dana (1994) develop equilibrium search models where firms have private information about their costs and consumers, who know the distribution of production costs as well as firms’ pricing strategies, learn about the distribution of prices as they search. Although this approach generates important insights (e.g., about the relationship between dispersion and unanticipated inflation), these informational requirements are clearly very strong. Another approach is that of Rauh (1997), where agents only know finitely many moments of the distribution of prices. Although this is weaker than knowing the entire distribution (all infinitely many moments), it is still not clear how agents obtain this information, which is contemporaneous.

In this paper, we pursue a dynamic approach to this issue. In particular, we show that “as if” knowledge of *last period’s* distribution of prices can be sufficient in the sense of convergent best response dynamics, where consumers’ reservation levels and firms’ profit-maximizing prices are best responses against the distribution of prices which prevailed in the previous period. This result is dramatically different from that in Hopkins and Seymour (2002), who model price dispersion as mixed-strategy equilibria (in our model, equilibrium is always in pure strategies) and show that such equilibria are generally unstable in the

class of models they examine. In our case, we can therefore interpret static search market equilibrium, where agents act as if they know the distribution of prices, as a long-run steady-state which can be arbitrarily closely approximated by a sequence of short-run “equilibria” in which agents behave as if they know last period’s distribution of prices, and whose ignorance vanishes in the limit.

The final issue concerns existence. In this literature, the pricing game involves a continuum of firms whose profits depend on their own prices as well as the *distribution* of prices in the market. Unfortunately, little seems to be known about games with this structure, so existence is a nontrivial problem. The main methodological innovation of the paper is our use of concepts and tools from the literature on supermodular games, including Milgrom and Roberts (1990), Topkis (1979, 1998), and Vives (1990, 1999).

We first derive necessary and sufficient conditions for the pricing game to exhibit strategic complementarities, which restrict how negative the slope of the density of search costs can be. In particular, a simple sufficient condition is that the density of search costs be at least weakly increasing over an initial segment of its domain. As explained below, this essentially implies that consumers have relatively high search costs, which seems consistent with recent empirical work by Hong and Shum (2006). It is also consistent with the following assessment of the relevant marketing literature

External information search is skewed toward limited search, with the greatest proportion of consumers performing little external search immediately prior to purchase. Surveys of shopping behavior have shown a significant percent of all durable purchases are made after visiting only one store... Measures of the use of personal and nonmarket sources also show somewhat limited levels of search... Based on six separate studies that span more than 30 years, two product categories, four services, and two countries... approximately half of the purchases are preceded by virtually no external information search; about one-third are associated with limited information search; and only 12 percent involve extensive information seeking prior to purchase.

Hawkins, Best, and Coney (1995, p. 451)

Although games with a continuum of players whose payoffs depend on the distribution of actions have not been analyzed in the literature on supermodular games, we show that several of the main results extend to equilibrium search models. In particular, when the game exhibits strategic complementarities, the existence question is easily resolved. In fact, we show that there exists a largest (high-price) and smallest (low-price) pure-strategy

Nash equilibrium, as in standard supermodular games. Moreover, existence is more than a theoretical curiosity, since we use it to prove our results on price dispersion: when firms are identical, the aforementioned extremal equilibria are symmetric (degenerate) and the Rob-type condition mentioned earlier then implies a unique symmetric equilibrium. It follows that the unique search market equilibrium is the Diamond equilibrium. A basic insight of the paper, therefore, is that price dispersion is inherently incompatible with strategic complementarities, where strategic variables tend to be set at comparable levels, whereas dispersion requires that some firms set high prices and others low ones.

If the pricing game exhibits strict strategic complementarities, we can also apply standard results from supermodular game theory on the stability of the best response dynamics, suitably adapted to the present context. Specifically, from any initial condition the best response dynamics approach the interval of strategy profiles determined by the smallest and largest equilibria, so a unique equilibrium is globally stable. An important special case is when the Diamond Paradox obtains and the unique Diamond equilibrium is globally stable as originally discovered by Diamond (1971). The literature on supermodular games is therefore utilized in a crucial way in proving all three main results of the paper on existence, price dispersion, and dynamics.

The plan for the rest of the paper is as follows. In section 2, we develop the model and characterize some of its essential features. In section 3, we derive conditions such that the pricing game exhibits strategic complementarities and establish the existence of search market equilibrium. We study price dispersion in section 4 and the stability of the best response dynamics in section 5. Section 6 concludes. All proofs not in the text are in the appendix.

2. The Model

The model is similar to those in Reinganum (1979), Carlson and McAfee (1983), Rob (1985), Bénabou (1993), and Rauh (2006). Let $I = [0, L]$ be the set of firms, $[0, M]$ the set of consumers, and $\theta = M/L$ the buyer-seller ratio, where $L, M > 0$. Let $C = [0, \bar{c}]$ be the set of feasible constant marginal costs and $m(i) \in C$ be the marginal cost of firm i .²

² All exogenously given functions such as $m : I \rightarrow C$ are assumed to be at least measurable.

Except for their costs, firms are otherwise identical. Let $\bar{m} = \sup_i m(i)$.

We assume all consumers have the same indirect utility function $U(p, y) = v(p) + y$, where p is price and y is income. Let $P = [0, \bar{p}]$ be the set of prices consumers are willing to pay, where $\bar{c} < \bar{p} < \infty$. Assume v is twice continuously differentiable on P (one-sided derivatives at the endpoints). The demand curve is therefore $x(p) = -v'(p)$, which is continuous on P .

Assumptions 1. (i) $x > 0$ on $[0, \bar{p})$ and $x = 0$ on $(\bar{p}, \infty)$. (ii) x is decreasing on P .³ (iii) For each $c \in C$, the monopoly problem

$$\max_{p \in [c, \bar{p}]} \pi^M(p, c) = (p - c)x(p) \quad (1)$$

has a unique solution $p_M(c) \in (c, \bar{p}]$ and $\pi_p^M \equiv \partial \pi^M / \partial p > 0$ on $[c, p_M(c))$.

These assumptions are general enough to cover most of the cases considered in the literature. For example, a common assumption is that demand is inelastic at one unit up to some maximum willingness to pay $\bar{p}$, so $x = 1$ on P . In that case, $p_M(c) = \bar{p}$ for all c and $\pi_p^M = 1$ on P . The assumptions also hold when demand is downward-sloping, twice differentiable on P , and not too convex, so that π^M is strictly concave. An example is linear demand $x(p) = a - bp$, where $a, b > 0$ and $\bar{p} = a/b$. It is easy to show that $p_M(c)$ is increasing in c .

In this context, a strategy profile is a (measurable) function f from firms I to prices P . Since no firm prices below its marginal cost or above its monopoly price, let $\mathcal{P}$ be the set of all strategy profiles such that $m(i) \leq f(i) \leq p_M(m(i))$ and $\mathcal{D}$ the set of all cumulative distribution functions (cdfs) induced by elements in $\mathcal{P}$. Furthermore, let F_m be the cdf of marginal costs (the cdf of prices induced by marginal cost pricing).

We make the standard assumptions that consumers search sequentially with recall and the first price quote is free.⁴ Let r be the lowest price the consumer has observed so

³ A function is *increasing* if $x > x'$ implies $f(x) \geq f(x')$ and *strictly increasing* when the latter inequality is strict. We define *decreasing* and *strictly decreasing* similarly.

⁴ The latter assumption has been relaxed in some recent papers; see Janssen and Moraga-González (2004), Rauh (2004), and Janssen, Moraga-González, and Wildenbeest (2005). In this paper, we assume the first price quote is free to avoid the necessity of keeping track of those consumers who may potentially drop out of the market.

far. As is well-known, the marginal benefit of another search is

$$\Gamma(r, F) = \int_0^r x(p)F(p) dp \quad (2)$$

[see Bénabou (1993)]. Let $\mathbf{R}$ and $\mathbf{R}_+$ denote the sets of real numbers and nonnegative real numbers, respectively. Proposition 1 below establishes some useful properties of Γ and introduces the parameter $\tilde{s}$.

Proposition 1. *For all $F \in \mathcal{D}$: (i) Γ is absolutely continuous on P with derivative $x(p)F(p)$ almost everywhere. (ii) If*

$$z(F) = \sup\{p \in \mathbf{R} \mid F(p) = 0\} \geq 0 \quad (3)$$

then $\Gamma = 0$ on the interval $[0, z(F)]$, $\Gamma > 0$ on $(z(F), \bar{p}]$, and strictly increasing on $[z(F), \bar{p}]$. (iii) $0 \leq \Gamma(r, F) \leq \tilde{s}$ for all $0 \leq r \leq p_M(\bar{m})$, where

$$\tilde{s} \equiv \int_0^{p_M(\bar{m})} x(r) F_m(r) dr. \quad (4)$$

Proposition 1(ii) concerns the basic shape of Γ , illustrated in Figure 1 below.

Figure 1 Goes Here

Note that Γ is zero between 0 and $z(F)$ and is positive and strictly increasing thereafter.

We now use the above result to define consumers' reservation levels and explain the significance of $\tilde{s}$. First of all, a consumer with search cost s stops searching at all prices r such that the marginal benefit $\Gamma(r, F)$ of another search is less than or equal to the marginal cost s , so her reservation level is

$$r(s, F) = \sup\{r \in P \mid \Gamma(r, F) \leq s\}. \quad (5)$$

In Figure 1, we see that the reservation level of the consumer with search cost s_1 is r_1 . Furthermore, the smallest possible reservation level is $z(F)$, which is the reservation level of the consumer with zero search cost (if any). All consumers with search costs $s \geq \Gamma(\bar{p}, F)$ have reservation level $\bar{p}$. Note that for each consumer, her reservation level exists and is unique since Γ is continuous and strictly increasing on $[z(F), \bar{p}]$.

A consumer whose reservation level is greater than or equal to almost all market prices searches exactly once with probability 1 and is therefore called an *inactive searcher*. Since consumers with reservation levels $r \geq p_M(\bar{m})$ are inactive searchers with search costs $s \geq \Gamma(p_M(\bar{m}), F)$, the proportion of inactive searchers is bounded from below by $1 - Q(\Gamma(p_M(\bar{m}), F))$, where Q is the cdf of consumers' search costs. According to the quotation in the introduction, the proportion of inactive searchers is often large in real-world markets. From (iii), $\Gamma(p_M(\bar{m}), F) \leq \tilde{s}$ for all $F \in \mathcal{D}$, so consumers with search costs $s \geq \tilde{s}$ are always inactive searchers.

Assume Q can be represented by a probability density function (pdf) q with support $S = [0, \bar{s}]$, where $0 < \bar{s} \leq \infty$. Note that $\bar{s}$ can be finite or infinite, so Assumptions 2 below are extremely mild.

Assumptions 2. (i) $q(s) > 0$ on $(0, \bar{s})$ and zero outside. (ii) q is bounded and continuous on $(0, \bar{s})$.

Note that we allow q to have potential discontinuities at $s = 0$ and $s = \bar{s}$ to accommodate the uniform distribution.

The cdf G of consumers' reservation levels is given by

$$G(r|F) = \begin{cases} Q(\Gamma(r, F)) & 0 \leq r < \bar{p} \\ 1 & r \geq \bar{p}, \end{cases} \quad (6)$$

with potential atom at $\bar{p}$ of size $1 - Q(\Gamma(\bar{p}, F))$. Using Proposition 1, we can represent G on $[0, \bar{p})$ by a density g defined almost everywhere by

$$g(r|F) = G'(r|F) = Q'(\Gamma(r, F))x(r)F(r) = q(\Gamma(r, F))x(r)F(r). \quad (7)$$

All firms face the same demand function, informally derived as follows. If firm i charges the price p , its potential customers have reservation levels $p \leq r(s, F) \leq \bar{p}$. Consumers with reservation level $\bar{p}$ are inactive searchers and are randomly and evenly distributed across all firms, so i expects to sell

$$\frac{M[1 - Q(\Gamma(\bar{p}, F))]}{L} x(p) = \theta x(p)[1 - Q(\Gamma(\bar{p}, F))] \quad (8)$$

to them. Let $Mg(r|F)dr$ be the mass of a consumer with reservation level r such that $p \leq r < \bar{p}$. Firm i competes for this consumer with all firms pricing less than or equal to r . Assuming $F(r) > 0$, expected sales to this consumer are

$$x(p) \frac{Mg(r|F)dr}{LF(r)} = \theta x(p) \frac{g(r|F)dr}{F(r)}. \quad (9)$$

Substituting from (7), we get

$$\theta x(p) q(\Gamma(r, F)) x(r) dr. \quad (10)$$

Summing over all consumers $p \leq r < \bar{p}$ and adding the potential atom at $\bar{p}$, the demand function is

$$D(p, F) = \theta x(p) N(p, F), \quad (11)$$

where

$$N(p, F) = \int_p^{\bar{p}} q(\Gamma(r, F)) x(r) dr + 1 - Q(\Gamma(\bar{p}, F)) \quad (12)$$

is the “number” of customers.

Proposition 2. *Demand D and the number of customers N are continuous and decreasing on P for all $F \in \mathcal{D}$. Moreover, N is differentiable on P except at most two points, with derivative $-q(\Gamma(p, F))x(p)$.*

From (11) it is clear that θ has no qualitative effect on our results, so from now on we set $\theta = 1$ for simplicity. Let $P_c = (c, p_M(c)]$. Firms’ profits are defined by

$$\pi(p, c, F) = (p - c)D(p, F) = \pi^M(p, c)N(p, F) \quad (13)$$

and the maximization problem of firm i is therefore⁵

$$\max_{p \in P_{m(i)}} \pi(p, m(i), F). \quad (14)$$

We now define the usual Nash equilibrium concept in this context.

⁵ Since π is continuous, it has a maximizer on $[c, p_M(c)]$. Since $p = c$ results in zero profits, $p_M(c)$ does at least as well and we can restrict attention to P_c .

Definitions 1. (i) A sequential search market is defined by (m, q, v) satisfying Assumptions 1 and 2. (ii) A search market equilibrium is a strategy profile f in $\mathcal{P}$ such that for all $i \in I$,

$$\pi(f(i), m(i), F_f) \geq \pi(p, m(i), F_f) \quad (15)$$

for all $p \in P$, where $F_f \in \mathcal{D}$ is induced by f .

Note that search market equilibria are defined for pure strategies only: we ignore mixed strategies throughout the paper. As Echenique and Edlin (2004) have shown, properly mixed strategy equilibria tend to be unstable in games with strategic complementarities, so there may be good reasons for doing so. Also note that optimal sequential search by consumers is incorporated into the demand function (11) by construction, so we do not explicitly mention it in the definition of Nash equilibrium.

3. Strategic Complementarities

From a game-theoretic perspective, the above equilibrium concept is somewhat unusual, since players' payoffs $\pi(p, m(i), F)$ depend on the cdf F of players' actions, as opposed to direct dependence on the strategy profile f itself. The existence issue is therefore a nontrivial one and we begin this section with a brief discussion of previous results.

In Reinganum (1979), consumers are completely identical with the same search cost and downward-sloping demand function and firms differ with respect to their production costs. Existence is not really an issue, since the equilibrium cdf of prices is directly induced by the cdf of firms' production costs. Rob (1985) considers the opposite polar case where firms are identical and consumers have perfectly inelastic demand ($\bar{p} = \infty$) and differ with respect to their search costs. Although his existence proof is ingenious, it does not take into account potentially downward-sloping demand and does not extend to the case of heterogeneous production costs. Carlson and McAfee (1983) also assume perfectly inelastic demand, but allow for bilateral heterogeneity in production and search costs, and provide an explicit solution for the equilibrium cdf of prices in the special case where the pdf of search costs is uniform. Bénabou (1990, 1993) generalizes Carlson and McAfee (1983)

to allow for downward-sloping demand and a broad class of search cost densities, but is unable to prove existence for reasons discussed in Bénabou (1990).

In Rauh (1997), consumers have a belief about the cdf of prices based on a forecast of finitely many of its moments. In equilibrium, consumers' forecasts about moments are correct, but their beliefs need not be. In this case, existence follows from the abstract game-theoretic result in Rauh (2003) [also see the extension in Yu and Zhu (2005)]. Finally, Rauh (2006) uses nonstandard analysis to prove existence in a general model with heterogeneities in consumers' demand functions, search costs, and firms' cost functions, for a broad class of demand and cost functions. Existence then follows from the abstract game-theoretic result in Khan and Sun (1999). Although this approach yields existence under very general conditions, it sheds little light on the structure of the equilibrium set, price dispersion, or dynamic stability properties.

In this paper, we study the subclass of equilibrium sequential search models with strategic complementarities. We define a binary relation on $\mathcal{D}$ as follows: $F \geq F'$ if $F(p) \leq F'(p)$ for all $p \in P$ (F stochastically dominates F' .) Under this relation, $\mathcal{D}$ is clearly a lattice.⁶ We write $F > F'$ if $F \geq F'$ and $F \neq F'$. We first investigate strategic complementarities in the form of increasing differences in $\log \pi$.⁷ Recall that solutions to a maximization problem are invariant to monotonic transformations of the objective function.

Definition 2. *A search market is (strictly) log-supermodular if $\log \pi(p, m(i), F)$ is well-defined and has (strictly) increasing differences on $P_{m(i)} \times \mathcal{D}$ for all i .*

Proposition 3 below provides a necessary and sufficient condition for (strict) log-supermodularity. Let $\underline{m} = \inf_i m(i)$.

Proposition 3. *If $Q(\tilde{s}) < 1$ then the search market is (strictly) log-supermodular iff the ratio $q(\Gamma(p, F))/N(p, F)$ is (strictly) decreasing on $\mathcal{D}$ for all $\underline{m} \leq p \leq p_M(\overline{m})$.*

⁶ Recall that a *partially ordered set* is a set X and a binary relation $\geq$ on X which is reflexive, transitive, and antisymmetric. A *lattice* $(X, \geq)$ is a partially ordered set such that $\inf\{x, x'\}$ and $\sup\{x, x'\}$ exist in X for all $x, x' \in X$. A lattice X is *complete* if $\sup S$ and $\inf S$ exist in X for all nonempty $S \subseteq X$.

⁷ Let X and Y be partially ordered sets. A function $f : X \times Y \rightarrow \mathbf{R}$ has *(strictly) increasing differences* on $X \times Y$ if $f(x, y) - f(x, y')$ is (strictly) increasing in x for all $y \geq y'$ ($y > y'$).

The condition $Q(\tilde{s}) < 1$ guarantees positive profits, so the (natural) logarithm of the latter is well-defined (see the proof).

The intuition for the result is as follows. Informally, the pricing game exhibits strategic complementarities when price increases by a firm's competitors induce the firm to raise its own price (see Proposition 5 below). Now, suppose firm i is charging the price p when a subset of its competitors raises their prices $F > F'$, increasing the number of firm i 's customers $N(p, F) > N(p, F')$. According to Proposition 3, a necessary and sufficient condition for log-supermodularity is that $q(\Gamma(p, F))$ does not increase too much. Since $\Gamma(p, F)$ is clearly decreasing in F , this implies q cannot have an overly negative slope. Why? If it did, there would be a relatively large mass of consumers with lower search costs whose reservation levels are too low to buy from firm i . The increase in the cdf of prices (in the sense of first-order stochastic dominance) raises their reservation levels, so that a sizable mass of consumers who formerly had a reservation level slightly below p would now have reservation level p . This would discourage firm i from raising its price, because it would then lose this large new mass of customers.

We now consider strategic complementarities in the form of increasing differences in π .

Definition 3. *A sequential search market is (strictly) supermodular if $\pi(p, m(i), F)$ has (strictly) increasing differences on $P_{m(i)} \times \mathcal{D}$ for all i .*

Let $F \geq F'$ and define

$$\Delta N(p, F, F') = N(p, F) - N(p, F'), \quad (16)$$

the change in the number of customers when F' increases to F at some given price p . A necessary and sufficient condition for (strict) supermodularity is that

$$\pi(p, m(i), F) - \pi(p, m(i), F') = \pi^M(p, m(i)) \Delta N(p, F, F') \quad (17)$$

be (strictly) increasing on $P_{m(i)}$. Using Proposition 2 to differentiate (17) with respect to price (except at most two points), we obtain

$$-x \Delta q \pi^M + \pi_p^M \Delta N \quad (18)$$

where

$$\Delta q = q(\Gamma(p, F)) - q(\Gamma(p, F')). \quad (19)$$

Since $\pi^M > 0$, we can re-arrange (18) to obtain (i) of Proposition 4 below.

Proposition 4. (i) *The search market is (strictly) supermodular iff*

$$\frac{\pi_p^M}{\pi^M} \Delta N \geq (>) x \Delta q \quad (20)$$

on $P_{m(i)}$ for all i and for all $F \geq F'$. (ii) In particular, if q is at least weakly increasing on $[0, \tilde{s}]$ then the game is supermodular. If q is strictly increasing on that interval, the game is strictly supermodular.

Condition (20) is similar to but different from that in Proposition 3, since (20) involves monopoly profits. If $p < \bar{p}$ then $x > 0$ and we can re-write (20) as

$$\frac{\pi_p^M}{x\pi^M} \Delta N \geq (>) \Delta q. \quad (21)$$

Since $\pi_p^M > 0$, Δq cannot be too large, so once again q cannot have an overly negative slope. The severity of this restriction compared to that in Proposition 3 depends on the size of the monopoly profit term multiplying ΔN .

Although the conditions in Proposition 3 and (20) are intuitive, they seem extremely difficult to check. In contrast, the condition in Proposition 4(ii) is readily ascertained given any q . We first note that the uniform density is constant on its support (hence weakly increasing), so the game is supermodular in that case. It is important to recognize, however, that (ii) only requires q to be increasing on $[0, \tilde{s}]$ and not on its entire domain. In particular, any density which is initially increasing and then decreasing, with $\tilde{s}$ less than the maximizer (mode) satisfies the condition. In that sense, (ii) is more of a condition on $\tilde{s}$ than on q itself.

Recall that consumers with search costs $s \geq \tilde{s}$ are always inactive searchers. The parameter $\tilde{s}$ therefore provides an upper bound for the search cost, conditional on F , which divides consumers into active and inactive searchers. If q has a standard hill- or bell-shape and is symmetric (not required for the Proposition), then (ii) requires that $\tilde{s}$

be less than the median search cost: the proportion of inactive searchers must be at least half the market, which is roughly consistent with the quotation in the introduction.

We now turn to individual firm comparative statics. We define a binary relation on $\mathcal{P}$ as follows: $f \geq f'$ if $f(p) \geq f'(p)$ for all $p \in P$. In the appendix, we show that $\mathcal{P}$ with this relation is a complete lattice (see Theorem A). Let ϕ be the best response correspondence defined by

$$\phi(i, f) = \arg \max_{p \in P_{m(i)}} \pi(p, m(i), F_f). \quad (22)$$

Given log-supermodularity or supermodularity, the proofs of the next two results are fairly standard.

Proposition 5. *If the game is either log-supermodular or supermodular then for each strategy profile f , the image $\phi(i, f) \subseteq P_{m(i)}$ of the best response correspondence is nonempty and compact, with largest $\bar{\phi}(i, f)$ and smallest $\underline{\phi}(i, f)$ elements, both measurable on I and increasing on $\mathcal{P}$.*

If best responses are always unique, it follows that each firm responds to price increases by its competitors by increasing its own price. Otherwise, we obtain weak monotone comparative statics in the sense that the extremal best responses increase, resulting in an upward shift in the set of best responses.

From now on, we say that the game exhibits *(strict) strategic complementarities* if the search market is either (strictly) log-supermodular or supermodular. Although the following result appears purely technical, it will play a crucial role in our analysis of price dispersion.

Theorem 1. *If the sequential search market exhibits strategic complementarities then the equilibrium set is nonempty and has a largest $\bar{f}$ and smallest $\underline{f}$ element.*

4. Price Dispersion

Price dispersion is ubiquitous; it has been observed everywhere from the modern Internet [Baye, Morgan, and Scholten (2006)] to the Turkish bazaar [Caglayan, Filiztekin, and Rauh (2006)]. It is the *raison d'être* for the equilibrium search literature.

Recall that the Diamond equilibrium is where almost all firms charge the unique monopoly price. Lemma 1 below shows that the Diamond equilibrium is the unique *symmetric* equilibrium when

$$q(0+) \equiv \lim_{s \rightarrow 0} q(s) \quad (23)$$

exists and is sufficiently small.⁸ As such, it represents a slight extension of the literature to the case of downward-sloping demand and heterogeneous production costs. Note that the result does not rely on strategic complementarities. Let F_{p_0} denote the cdf of prices that assigns probability 1 to some $p_0 \geq 0$.

Lemma 1. *Assume $q(0+)$ exists. (i) If demand x is perfectly inelastic at one unit on P and $q(0+) < 1/\bar{p}$ then $F_{\bar{p}}$ is the unique symmetric equilibrium. (ii) If firms are identical with marginal cost c and*

$$q(0+) < \frac{\pi_p^M}{x\pi^M} \quad (24)$$

for all $c < p < p_M(c)$ then $F_{p_M(c)}$ is the unique symmetric equilibrium.

In (i), the Diamond equilibrium obtains no matter how heterogeneous production and search costs are. In (ii), it obtains when demand is properly downward-sloping and no matter how heterogeneous search costs are. In both cases, there are no more symmetric equilibria, although we have yet to rule out dispersed (asymmetric) equilibria.

The intuition for uniqueness is as follows. Consider case (ii) and suppose almost all firms are charging p_0 , where $c < p_0 < p_M(c)$. If a firm were to raise its price above p_0 , its monopoly profits would increase [see Assumptions 1 and (13)] but it would lose its customers with low search costs. When $q(0+)$ is relatively small, the former effect dominates and F_{p_0} cannot be an equilibrium. The result therefore confirms and extends the conventional wisdom about the importance of the behavior of the density of search costs near zero.

We now come to the main result of this section.

⁸ Recall that q may have a discontinuity at zero (e.g., the uniform distribution) so $q(0+)$ can be different from $q(0)$.

Theorem 2. *If the search market exhibits strategic complementarities, firms are identical, and $q(0+)$ is sufficiently small in the sense of Lemma 1, then the Diamond Paradox obtains: the Diamond equilibria in Lemma 1 are the unique search market equilibria overall for their respective cases.*

Proof.

Consider case (i) or (ii) in Lemma 1. According to that lemma, the Diamond equilibrium is the unique symmetric equilibrium. By Theorem 1, the set of search market equilibria has a largest $\bar{f}$ and smallest $\underline{f}$ element. In the case of $\bar{f}$, the equilibrium involves firms making their largest best response $\bar{f}(i)$ against $\bar{f}$ (see the proof of Theorem 1). Since firms are identical, their largest best response is the same, and $\bar{f}$ is a symmetric equilibrium. Since $\underline{f}$ is also a symmetric equilibrium, $\bar{f} = \underline{f}$ and equilibrium is unique: the unique equilibrium is the Diamond equilibrium. ■

This result illustrates the power of the theory of supermodular games for the study of price dispersion. Given the existing literature, lemma 1 is not that surprising: when firms are identical and a Rob (1985)-type condition on $q(0+)$ holds, the Diamond equilibrium is the unique single-price equilibrium. What *is* surprising is the effect of combining that result with strategic complementarities: in that case, the Diamond equilibrium is the unique equilibrium overall, and no amount of heterogeneity in search costs can overcome the Diamond Paradox.

It follows that to induce dispersion, we need some other source of heterogeneity. The following result shows that a simple Reinganum-type condition is sufficient for *all* equilibria to be dispersed (existence is assured by Theorem 1). Note that this condition rules out perfectly inelastic demand since we have assumed $\bar{c} < \bar{p}$.

Lemma 2. *Let I_R be the set of all i such that there exists $I' \subseteq I$ of positive measure such that $p_M(m(i)) < m(i')$ for all $i' \in I'$. If I_R has positive measure then degenerate equilibria do not exist.*

In this case, dispersion is not a purely informational phenomenon but is instead driven

by imperfect information, downward-sloping demand, and heterogeneous production costs. As such, it is at least in part a *technological* phenomenon.

We now compare the above results with Rob (1985), where firms are all identical and consumers have perfectly inelastic demand and differ with respect to their search costs. First of all, the assumption that consumers have infinite willingness to pay ($\bar{p} = \infty$) eliminates the Diamond equilibrium *ex ante*. Without it, other single-price equilibria do not exist under the mild assumption that

$$Q(s) < \frac{sq(0+)}{1 + sq(0+)} \quad (25)$$

for at least one $s > 0$ [Rob (1985, theorem 1)]. Assuming the latter, the existence of price dispersion boils down to plain existence, which is again guaranteed under mild conditions [Rob (1985, lemma 4 and Theorem 2)]. The bottom line is that dispersion obtains under mild assumptions and is robust in that sense.

In contrast, when demand is inelastic with finite willingness to pay or is downward-sloping, the above methodology breaks down. Once the Diamond equilibrium is in the equilibrium set, existence is no longer an issue and the existence of price dispersion no longer equivalent to plain existence. Moreover, the Diamond Paradox obtains over a robust and natural region of model-space: the subset of equilibrium search models with strategic complementarities.

5. Dynamics

In equilibrium, consumers and firms act as if they know the cdf of prices, which requires a substantial amount of information and computational power. First of all, the model (m, q, v) must be common knowledge: agents must know consumers' search costs, consumer demand x , and firms' production costs. Furthermore, agents must be able to actually compute the equilibrium cdf of prices, something the theoretical literature itself has been unable to do except in certain special cases [see Carlson and McAfee (1983)].

In this paper, we assume agents are myopic and have enough information to re-create the cdf of prices which prevailed *in the previous period*. Although real-world agents may not even possess this information, these assumptions are nevertheless orders of magnitude

weaker than those above. Given last period's cdf of prices, consumers can calculate their reservation levels knowing only their own demand and search cost. To compute their profit-maximizing prices, firms still need to know consumers' demand and search costs, but they need not know other firms' costs or be able to compute the equilibrium cdf of prices. We assume given some initial condition; the uniform distribution is a natural choice.

A *best response orbit* is therefore a sequence $\{f_n\}$ of strategy profiles such that $f_{n+1}(i) \in \phi(i, f_n)$ for all $i \in I$ and $n \geq 0$ for some initial condition $f_0 \in \mathcal{P}$. In other words, consumers' reservation levels and firms' profit-maximizing prices are best responses against last period's cdf of prices.⁹ Given two strategy profiles a and b , we can define an interval in $\mathcal{P}$ as follows:

$$[a, b] = \{f \in \mathcal{P} \mid a \leq f \leq b\}. \quad (26)$$

We say that a sequence $\{f_n\}$ of strategy profiles *approaches* $[a, b]$ if there exist two sequences $\{\underline{f}_n\}$ and $\{\bar{f}_n\}$ of strategy profiles such that $\underline{f}_n \leq f_n \leq \bar{f}_n$ for all $n \geq 0$ and $\underline{f}_n \uparrow a$ and $\bar{f}_n \downarrow b$ almost uniformly.¹⁰ In other words, $\{f_n\}$ is being “squeezed” towards $[a, b]$ from above and below by two sequences in $\mathcal{P}$ converging monotonically and almost uniformly to its endpoints. Let

$$\begin{aligned} \mathcal{P}^+ &= \{f \in \mathcal{P} \mid f \geq \bar{\Psi}(f)\} \\ \mathcal{P}^- &= \{f \in \mathcal{P} \mid f \leq \underline{\Psi}(f)\}, \end{aligned} \quad (27)$$

where $\bar{\Psi} : \mathcal{P} \rightarrow \mathcal{P}$ is defined by $f \mapsto \bar{\phi}(i, f)$ and similarly for $\underline{\Psi}$.

Milgrom and Roberts (1990) and Vives (1990) have shown that supermodular games with finitely many players have nice stability properties. Theorem 3 below shows that, to some extent, this extends to equilibrium search models with a continuum of firms. Given the above definitions, the proof involves minor adaptations of the argument for theorem 5.1 in Vives (1990, p. 313) to account for the infinite-dimensional and measure-theoretic aspects of the current context.

⁹ Recall that optimal sequential search is built into firms' demand function; see (2) and (12). Moreover, f_n enters $\phi(i, f_n)$ through the induced cdf F_{f_n} of prices; see (22). The notation is therefore slightly misleading, since it suggests that agents act as if they know f_n , when in fact they act as if they know F_{f_n} .

¹⁰ Let f and f_n for all $n \geq 1$ be measurable functions $I \rightarrow \mathbf{R}$ and λ denote Lebesgue measure on $\mathbf{R}$. We say that $f_n \rightarrow f$ *almost uniformly* if for all $\epsilon > 0$ there exists $I' \subseteq I$ such that $\lambda(I') < \epsilon$ and $f_n \rightarrow f$ uniformly on the complement of I' in I . See Folland (1999, p. 60). We write $f_n \uparrow f$ if $f_n \leq f_{n+1}$ for all $n \geq 1$ and $f_n \rightarrow f$ almost uniformly. The definition for $f_n \downarrow f$ is similar.

Theorem 3. *Assume strict strategic complementarities and that if the cdf of prices is the same in periods n and $n + 1$ then firms set the same prices at $n + 2$ as they did in $n + 1$.*

(i) Every best response orbit approaches $[\underline{f}, \bar{f}]$, where $\underline{f}$ is the smallest search market equilibrium and $\bar{f}$ is the largest.

(ii) For any best response orbit $\{f_n\}$ with initial condition in $\mathcal{P}^+$ ($\mathcal{P}^-$), there exists a search market equilibrium $f \in E$ such that $f_n \downarrow f$ ($f_n \uparrow f$).

(iii) If equilibrium is unique, then every best response orbit approaches it.

According to (i), all best response orbits approach the interval $[\underline{f}, \bar{f}]$ in $\mathcal{P}$ defined by the smallest and largest equilibria. It follows that if the best response dynamics exhibit any regularities (limiting behaviors such as convergence to a fixed point or periodic cycle), then such regularities will either be in $[\underline{f}, \bar{f}]$ or arbitrarily close. In (ii), we show that best response dynamics starting from initial conditions in $\mathcal{P}^+$ or $\mathcal{P}^-$ converge to search market equilibria. In particular, in the proof we show that the best response orbit starting from monopolistic expectations $p_M(m(i)) \in \mathcal{P}^+$ converges to $\bar{f}$ and that starting from perfectly competitive expectations $m(i) \in \mathcal{P}^-$ converges to $\underline{f}$, so the result is not vacuous. In the long-run, agents act as if they know the equilibrium cdf of prices and such behavior can be approximated arbitrarily closely by short-run “equilibria” where consumers’ reservation levels and firms’ prices are best responses against last period’s distribution of prices. In (iii), we have the strong result that a unique equilibrium is approached from all initial conditions (global stability), where “approach” is a strong form of convergence involving monotonic almost uniform squeezing from above and below. In particular, when the Diamond Paradox obtains, the Diamond equilibrium is globally stable as discovered by Diamond (1971).

6. Conclusion

In this paper, we applied concepts and tools from the theory of supermodular games to the equilibrium search literature with sequential search. We considered the class of equilibrium search models in Carlson and McAfee (1983) and Bénabou (1993) with bilateral heterogeneities in consumers’ search costs and firms’ production costs and potentially downward-sloping demand with finite willingness to pay. We identified necessary and

sufficient conditions for the pricing game to exhibit strategic complementarities, which essentially restrict how negative the slope of the density of search costs can be. In particular, a sufficient condition is that the density be at least weakly increasing over an initial segment of its domain. This implies consumers have relatively high search costs, which seems consistent with empirical evidence.

Given strategic complementarities, we proved the existence of search market equilibrium. Indeed, there exists a largest and smallest equilibrium as in standard supermodular games. We then showed that the Diamond Paradox obtains when firms are identical, a Rob (1985)-type condition holds with respect to the behavior of the density of search costs near zero, and the pricing game exhibits strategic complementarities. According to conventional wisdom, distilled from the seminal Rob (1985) paper where demand is perfectly inelastic with infinite willingness to pay, price dispersion can arise solely from imperfect information and heterogeneous search costs under mild assumptions. In contrast, in our model the Diamond Paradox is immune to the distribution of search costs within the class that generates strategic complementarities. A basic insight of the paper is that price dispersion is therefore inherently incompatible with strategic complementarities, where strategic variables tend to be set at comparable levels. On the other hand, a simple Reinganum (1979)-type condition on firms' production costs ensures that all equilibria are dispersed. Finally, we show that a major criticism of the literature, that agents act as if they know the distribution of prices, can be justified in the sense of convergent best response dynamics.

Appendix

Proof of Proposition 1

We first note that $\Gamma : P \times \mathcal{D} \rightarrow \mathbf{R}_+$ defined in (2) is well-defined since the integrand is measurable and dominated by an integrable function: $0 \leq x(p)F(p) \leq x(p)$. (i) follows from theorem 5.5 and proposition 6.5 in Haaser and Sullivan (1991, p. 237, 240). (ii) is obvious. Since F first-order stochastically dominates F_m for all $F \in \mathcal{D}$,

$$0 \leq \Gamma(r, F) \leq \Gamma(p_M(\overline{m}), F) = \int_0^{p_M(\overline{m})} x(p) F(p) dp$$

$$\leq \int_0^{p_M(\bar{m})} x(p) F_m(p) dp \equiv \tilde{s} \quad (A.1)$$

for all $0 \leq r \leq p_M(\bar{m})$, so (iii) follows. ■

Before proving Proposition 2, we briefly return to (7). This reveals the form of the pdf g for G on $[0, \bar{p})$, but we have not shown that G is absolutely continuous on that interval. The latter follows easily from the assumption that Q is absolutely continuous and $\Gamma(r, F)$ is increasing and absolutely continuous by Proposition 1.

Proof of Proposition 2

Recall that the product of a bounded measurable function and an integrable function is integrable. Now, $q(\Gamma(r, F))$ is bounded and continuous in r except possibly at $r = z(F)$ and at the point where $\Gamma = \bar{s}$ if $\bar{s}$ is finite. It follows that $q(\Gamma(r, F))$ is bounded measurable. Since $x(r)$ is integrable, the integral in (12) is well-defined. The rest follows from theorem 5.5 and proposition 6.5 in Haaser and Sullivan (1991, p. 237, 240). ■

Proof of Proposition 3

Since $Q(\tilde{s}) < 1$, the proportion of inactive searchers is positive for all $F \in \mathcal{D}$. Since these consumers are spread evenly across all firms, $N(p, F)$ is positive for all $F \in \mathcal{D}$. Moreover, π^M is positive on $P_{m(i)}$, so π is positive and the logarithm in the statement of the result is well-defined. Let $F > F'$ in $\mathcal{D}$. We must show that

$$\log \pi(p, m(i), F) - \log \pi(p, m(i), F') = \log N(p, F) - \log N(p, F') \quad (A.2)$$

is (strictly) increasing on $P_{m(i)}$. By Proposition 2, we can differentiate (A.2) except at most two points to get

$$-\frac{q(\Gamma(p, F))x(p)}{N(p, F)} + \frac{q(\Gamma(p, F'))x(p)}{N(p, F')} \geq 0 \quad (> 0), \quad (A.3)$$

which reduces to

$$\frac{q(\Gamma(p, F'))}{N(p, F')} \geq (>) \frac{q(\Gamma(p, F))}{N(p, F)}. \quad (A.4)$$

This must hold for all $p \in P_{m(i)}$ and all i , so it must hold on $[\underline{m}, p_M(\overline{m})]$. ■

Proof of Proposition 4

We prove (ii). Since π^M is strictly increasing on $P_{m(i)}$, from (17) it is sufficient to prove that $\Delta N(p, F, F')$ is increasing. The latter holds iff

$$\int_p^{\bar{p}} [q(\Gamma(r, F)) - q(\Gamma(r, F'))] x(r) dr \quad (A.5)$$

is increasing. This requires

$$q(\Gamma(p, F)) \leq q(\Gamma(p, F')) \quad (A.6)$$

for all $p \in P_{m(i)}$. Since Γ is decreasing on $\mathcal{D}$, q must be increasing. By Proposition 1, $0 \leq \Gamma(p, F'') \leq \tilde{s}$ for all $F'' \in \mathcal{D}$, so we require q to be increasing on $[0, \tilde{s}]$.

To prove strictly increasing differences, let $F > F'$ in $\mathcal{D}$. We must now show that (A.5) is strictly increasing on $P_{m(i)}$, which requires (A.6) to be strict. Since $F > F'$, there exists $0 \leq p_0 < \bar{p}$ such that $F(p_0) < F'(p_0)$. This holds on an open neighborhood to the right of p_0 because F and F' are everywhere right-continuous. Since $x > 0$ on that open neighborhood, we have $\Gamma(p, F) < \Gamma(p, F')$, so it is sufficient that q be strictly increasing on $[0, \tilde{s}]$. ■

The following result is similar to lemma 6.1 in Vives (1990, p. 315). A lattice X is *conditionally complete* if $\sup S$ and $\inf S$ exist in X for all nonempty and bounded $S \subseteq X$. For example, $\mathbf{R}$ is conditionally complete.

Formally, we identify strategy profiles which are equal almost everywhere with respect to Lebesgue measure, so $\mathcal{P}$ is the space of all equivalence classes of such profiles.

Theorem A. *$\mathcal{P}$ is a complete lattice.*

Proof. Clearly, $\mathcal{P}$ is a lattice. Let $S \subseteq \mathcal{P}$ be nonempty. Welland (1964, p. 267-8) shows that the space L_1 of equivalence classes of Lebesgue integrable functions $I \rightarrow \mathbf{R}$ is conditionally complete. Each $f \in \mathcal{P}$ is bounded and measurable, so $S \subseteq L_1$. Recall that an increasing real-valued function on a compact interval is bounded and a.e. differentiable,

hence Riemann and Lebesgue integrable [Haaser and Sullivan (1991, theorem 3.7, p. 228)]. A composition of measurable functions is measurable, so $p_M(m(i))$ is bounded, measurable, and integrable on I . Since S is bounded from above by $p_M \circ m \in L_1$ and from below by $m \in L_1$ and L_1 is conditionally complete, it follows that $\sup S$ and $\inf S$ exist in L_1 . Since $m \leq \inf S \leq \sup S \leq p_M \circ m$, we have $\inf S, \sup S \in \mathcal{P}$ and we are done. ■

Proof of Proposition 5

Fix $i \in I$. Each $f \in \mathcal{P}$ induces an $F \in \mathcal{D}$. For all $F \in \mathcal{D}$, π is continuous and trivially log-supermodular or supermodular on $P_{m(i)}$. By theorem 2.4 and its corollary in Vives (1999, p. 30), $\phi(i, f)$ is nonempty and compact, with largest $\bar{\phi}(i, f)$ and smallest $\underline{\phi}(i, f)$ elements. Now fix $f \in \mathcal{P}$ and the $F_f \in \mathcal{D}$ it induces. By the Measurable Maximum Theorem in Aliprantis and Border (1999, p. 570), the best response correspondence ϕ is measurable on I . Applying the Measurable Maximum Theorem again, with p as the objective function and ϕ as the constraint correspondence, we see that $\bar{\phi}$ is measurable on I . A similar argument with $-p$ as the objective function shows that $\underline{\phi}$ is measurable on I . The fact that the extremal best responses are increasing on $\mathcal{P}$ follows from $f \geq f' \Rightarrow F_f \geq F_{f'}$, increasing differences on $P_{m(i)} \times \mathcal{D}$, and theorem 2.4 and its corollary in Vives (1999, p. 30), which completes the proof. ■

Proof of Theorem 1

According to Proposition 6, $\bar{\phi}(i, f)$ is measurable on I and the map $\bar{\Psi} : \mathcal{P} \rightarrow \mathcal{P}$ defined by $f \mapsto \bar{\phi}(i, f)$ is increasing. We can therefore apply the Tarski fixed point theorem [theorem 2.2 in Vives (1999, p. 20)] to obtain the largest fixed point $\bar{f}$ of $\bar{\Psi}$ satisfying

$$\bar{f} = \sup\{f \in \mathcal{P} \mid \bar{\Psi}(f) \geq f\}. \quad (A.7)$$

Let f be any Nash equilibrium. Since $\bar{\phi}(i, f) \geq f(i)$ for all $i \in I$, it follows that $\bar{\Psi}(f) \geq f$ and hence $\bar{f} \geq f$. The proofs for $\underline{f}$ are similar. ■

Proof of Lemma 1

It is obvious that $F_{\bar{p}}$ and $F_{p_M(c)}$ are equilibria in their respective cases. To prove (ii), let $c \leq p_0 < p_M(c)$. We first derive the expression for the right-hand derivative of π at p_0 . By Proposition 2, N is decreasing on P for all $F \in \mathcal{D}$, so it is everywhere differentiable from the right. For $\epsilon > 0$ sufficiently small, $p_0 < p_0 + \epsilon < p_M(c) \leq \bar{p}$ and

$$\frac{N(p_0 + \epsilon, F_{p_0}) - N(p_0, F_{p_0})}{\epsilon} = - \frac{\int_{p_0}^{p_0 + \epsilon} q \left(\int_{p_0}^r x(p) dp \right) x(r) dr}{\epsilon}. \quad (\text{A.8})$$

Next, we note that

$$0 < \int_{p_0}^{p_0 + \epsilon} x(p) dp < \bar{s} \quad (\text{A.9})$$

for $\epsilon > 0$ sufficiently small. In (A.8), the numerator and denominator $\rightarrow 0$ as $\epsilon \rightarrow 0$, so we apply L'Hospital's Rule to get

$$\lim_{\epsilon \rightarrow 0} \frac{N(p_0 + \epsilon, F_{p_0}) - N(p_0, F_{p_0})}{\epsilon} = - \lim_{\epsilon \rightarrow 0} q \left(\int_{p_0}^{p_0 + \epsilon} x(p) dp \right) x(p_0 + \epsilon). \quad (\text{A.10})$$

In differentiating the numerator, we used the assumption that q is continuous on $(0, \bar{s})$ and (A.9). Since x is continuous and by hypothesis $q(0+)$ exists,

$$\lim_{\epsilon \rightarrow 0} q \left(\int_{p_0}^{p_0 + \epsilon} x(p) dp \right) x(p_0 + \epsilon) = q(0+)x(p_0). \quad (\text{A.11})$$

We now take the right-hand derivative of π at p_0 to get

$$-\pi^M q(0+)x(p_0) + N(p_0, F_{p_0})\pi_p^M. \quad (\text{A.12})$$

Since consumers are evenly spread across firms, $N(p_0, F_{p_0}) = 1$. The expression in (A.12) is therefore positive if

$$-\pi^M q(0+)x(p_0) + \pi_p^M > 0. \quad (\text{A.13})$$

We have $\pi_p^M(p_0, c) > 0$ since $c \leq p_0 < p_M(c)$. If $p_0 = c$, then $\pi^M = 0$ and (A.13) holds. If $c < p_0 < p_M(c)$, (A.13) boils down to (24). Hence, almost all firms prefer to raise price. The proof for (i) is similar but easier. ■

Proof of Lemma 2

Suppose F_{p_0} is an equilibrium cdf of prices for some $p_0 \in P$. We must therefore have $p_0 \in [m(i), p_M(m(i))]$ for almost all i . Take any $i \in I_R$. Since I' has positive measure, we must have $p_M(m(i)) < m(i') \leq p_0$ for almost all $i \in I'$, so i cannot be making a best response. Since I_R has positive measure, we are done. ■

A correspondence ϕ from a partially ordered set X into a lattice Y is *strongly increasing* if $x > x'$ implies $y \geq y'$ for all $y \in \phi(x)$ and $y' \in \phi(x')$. In other words, a correspondence is strongly increasing iff all its selections are increasing.

Proof of Theorem 3

We first prove (ii). Let $f_0 \in \mathcal{P}^+$. If firms always choose their largest best response, we have a best response orbit $\{\bar{f}_n\}_{n=0}^\infty$ defined by $\bar{f}_{n+1} = \bar{\Psi}(\bar{f}_n)$ (where $\bar{f}_0 = f_0$). By definition, $\bar{f}_1 = \bar{\Psi}(f_0) \leq f_0$. Since $\bar{\Psi}$ is increasing, $\bar{f}_{n+1} \leq \bar{f}_n$ for all $n \geq 0$. If firms always choose their smallest best response, $\{\underline{f}_n\}$ is defined by $\underline{f}_{n+1} = \underline{\Psi}(\underline{f}_n)$. Since $\underline{f}_1 = \underline{\Psi}(f_0) \leq \bar{\Psi}(f_0) \leq f_0$ and $\underline{\Psi}$ is also increasing, $\underline{f}_{n+1} \leq \underline{f}_n$ for all $n \geq 0$. Now take any best response orbit $\{f_n\} \subseteq \mathcal{P}$ with initial condition f_0 . Since $f_1(i) \in \phi(i, f_0)$ for all i , $f_1 \leq \bar{\Psi}(f_0) \leq f_0$. Let F_n be the cdf induced by f_n for all $n \geq 0$. If $f_1 = f_0$ for almost all i , then $F_1 = F_0$ and $f_2 = f_1$ by hypothesis. Otherwise, $F_1 < F_0$ and by hypothesis π has strictly increasing differences on $P_i \times \mathcal{D}$ for all i . By theorem 2.3(iv) in Vives (1999, p. 26), firms' best response correspondences are strongly increasing on $\mathcal{D}$. Hence, $f_2 \leq f_1$ and $f_{n+1} \leq f_n$ for all $n \geq 0$ follows by induction. The best response orbit is therefore decreasing in all cases.

Let $\{f_n\}$ be any decreasing best response orbit. By the monotone convergence theorem [Lang (1993, Theorem 5.5, p. 139)], there exists an integrable $f : I \rightarrow \mathbf{R}$ such that $f_n(i) \rightarrow f(i)$ almost everywhere on I . Since $m(i) \leq f_n(i) \leq p_M(m(i))$ for all i and $n \geq 0$, the same holds for f almost everywhere. By Egoroff's theorem [Folland (1999, p. 62)], the convergence $f_n \rightarrow f$ is almost uniform. Let E denote the set of search market equilibria. We now show that $f \in E$, once we have fixed it on a set of measure zero. First, fix any $i \in I$ in the subset of I of full measure such that $f_n(i) \rightarrow f(i)$. Let $\mathcal{D}$ be endowed with the

topology of weak convergence. According to Dudley (2002, p. 288, 292, 295), $F_n \rightarrow F_f$ weakly, where F_f is the cdf induced by f . Furthermore, π is continuous on $P_i \times \mathcal{D}$ by Proposition 3 in Rauh (2006). Since $f_n(i) \rightarrow f(i)$, $F_n \rightarrow F_f$, and $f_n(i) \in \phi(i, f_{n-1})$ for all $n \geq 1$, we have $f(i) \in \phi(i, f)$ because π is continuous. We are then free to alter f for the other firms (who comprise a set of measure zero) so that they are making best responses against f . After doing so, f is a Nash equilibrium. The proof for $f \in \mathcal{P}^-$ is similar.

We now prove (i). Let $\{\bar{f}_n\}$ be defined as before, starting from $\bar{f}_0 = p_M \circ m$ and $\{\underline{f}_n\}$ starting from $\underline{f}_0 = m$. Clearly, $\bar{f}_0 \in \mathcal{P}^+$ and $\underline{f}_0 \in \mathcal{P}^-$. From (ii), there exists $\bar{f} \in E$ ($\underline{f} \in E$) such that $\bar{f}_n \downarrow \bar{f}$ ($\underline{f}_n \uparrow \underline{f}$) and convergence is also pointwise almost everywhere. We now show that $\bar{f}$ is the largest search market equilibrium, recalling that we identify functions which differ at most on a set of measure zero. Fix any $f \in E$. Clearly, $\bar{f}_0 = p_M \circ m \geq f$. Moreover, $\bar{f}_n \geq f$ implies

$$\bar{f}_{n+1} = \bar{\Psi}(\bar{f}_n) \geq \bar{\Psi}(f) \geq f, \quad (\text{A.14})$$

so $\bar{f}_n \geq f$ for all $n \geq 0$. It follows that $\bar{f} \geq f$ almost everywhere, so we are done. A similar argument shows that $\underline{f}$ is the smallest search market equilibrium. Now take any best response orbit $\{f_n\}$ with initial condition f_0 . Clearly, $\underline{f}_0 \leq f_0 \leq \bar{f}_0$. Furthermore,

$$\underline{f}_1 = \underline{\Psi}(\underline{f}_0) \leq \underline{\Psi}(f_0) \leq f_1 \leq \bar{\Psi}(f_0) \leq \bar{\Psi}(\bar{f}_0) = \bar{f}_1. \quad (\text{A.15})$$

By induction, $\underline{f}_n \leq f_n \leq \bar{f}_n$, so we are done. ■

References

C.D. Aliprantis, K.C. Border, *Infinite Dimensional Analysis*, Springer-Verlag, New York, 1999.

M.R. Baye, J. Morgan, P. Scholten, *Information, search, and price dispersion*, *Handbook on Economics and Information Systems* (Elsevier, forthcoming).

R. Bénabou, “Search Market Equilibrium, Bilateral Heterogeneity, and Repeat Purchases,” MIT mimeo, 1990.

- R. Bénabou, Search market equilibrium, bilateral heterogeneity, and repeat purchases, *J. Econ. Theory* 60 (1993), 140-158.
- R. Bénabou, R. Gertner, Search with learning from prices: does increased uncertainty lead to higher markups? *Rev. Econ. Stud.* 60 (1993), 69-94.
- M. Caglayan, A. Filiztekin, M.T. Rauh, Inflation, price dispersion, and market structure, working paper, (2006).
- J. Carlson, P. McAfee, Discrete equilibrium price dispersion, *J. Polit. Econ.* 91 (1983), 480-493.
- J.D. Dana, Jr., Learning in an equilibrium search model, *Int. Econ. Rev.* 35 (1994), 745-71.
- P.A. Diamond, A model of price adjustment, *J. Econ. Theory* 3 (1971), 156-68.
- R.M. Dudley, *Real Analysis and Probability*, Cambridge University Press, Cambridge, 2002.
- F. Echenique, A. Edlin, Mixed equilibria are unstable in games of strategic complements, *J. Econ. Theory* 118 (2004), 61-79.
- G.B. Folland, *Real Analysis*, John Wiley and Sons, Inc., New York, 1999.
- N.B. Haaser, J.A. Sullivan, *Real Analysis*, Dover, New York, NY, 1991.
- D.I. Hawkins, R.J. Best, K.A. Coney, *Consumer Behavior: Implications for Marketing Strategy*, Irwin, Chicago, 1995.
- H. Hong, M. Shum, Using price distributions to estimate search costs, *RAND J. Econ.* (forthcoming).
- E. Hopkins, R.M. Seymour, The stability of price dispersion under seller and consumer learning, *Int. Econ. Rev.* 43 (2002), 1157-1190.

- M.C.W. Janssen, J.L. Moraga-González, Strategic pricing, consumer search and the number of firms, *Rev. Econ. Stud.* 71 (2004), 1089-1118.
- M.C.W. Janssen, J.L. Moraga-González, M.R. Wildenbeest, Truly costly sequential search and oligopolistic pricing, *Intl. J. Ind. Org.* 23 (2005), 451-66.
- M.A. Khan, Y.N. Sun, Non-cooperative games on hyperfinite Loeb spaces, *J. Math. Econ.* 31 (1999), 455-492.
- S. Lang, *Real and Functional Analysis*, Springer-Verlag, New York, 1993.
- S.A. Lippman, J.J. McCall, The economics of uncertainty, in: K.J. Arrow and M.D. Intriligator, eds., *Handbook of Mathematical Economics*, Vol. I (North-Holland, Amsterdam) (1981), p. 211-227.
- P. Milgrom, J. Roberts, Rationalizability, learning, and equilibrium in games with strategic complementarities, *Econometrica* 58 (1990), 1255-77.
- M.T. Rauh, A model of temporary search market equilibrium, *J. Econ. Theory* 77 (1997), 128-153.
- M.T. Rauh, Non-cooperative games with a continuum of players whose payoffs depend on summary statistics, *Econ. Theory* 21 (2003), 901-906.
- M.T. Rauh, Wage and price controls in the equilibrium sequential search model, *Europ. Econ. Rev.* 48, (2004), 1287-1300.
- M.T. Rauh, Nonstandard foundations of equilibrium search models, *J. Econ. Theory*, forthcoming.
- J.F. Reinganum, A simple model of equilibrium price dispersion, *J. Polit. Econ.* 87 (1979), 851-858.
- R. Rob, Equilibrium price distributions, *Rev. Econ. Stud.* 81 (1985), 1283-1308.

M. Rothschild, Searching for the lowest price when the distribution of prices is unknown, J. Polit. Econ. 82 (1974), 689-711.

D.M. Topkis, Equilibrium points in nonzero-sum n -person submodular games, SIAM J. Cont. Opt. 17 (1979), 773-787.

D.M. Topkis, Supermodularity and Complementarity, Princeton University Press, Princeton, NJ, 1998.

X. Vives, Nash equilibrium with strategic complementarities, Journal of Mathematical Economics 19 (1990), 305-21.

X. Vives, Oligopoly Pricing, MIT Press, London, England, 1999.

R. Welland, On Kothe spaces, Trans. Amer. Math. Soc. 112 (1964), 267-277.

D. Williams, Probability with Martingales, Cambridge University Press, Cambridge, England, 1991.

Figure 1