

Timing of Effort and Reward: Three-Sided Moral Hazard in a Continuous-Time Model

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This paper studies a three-sided moral hazard problem with one agent exerting up-front effort and two agents exerting ongoing effort in a continuous-time model. The agents' efforts jointly affect the probability of survival and thus the expected cash flow of the project. In the optimal contract, the timing of payments reflects the timing of effort: payments for up-front effort precede payments for ongoing effort. Several patterns are possible for the cash allocation between the two agents with ongoing effort. In one case, where the two agents face equally severe moral hazard, they share the cash flow equally at each point of time. In another case, where the two agents have different severities of moral hazard, their payments are sequential. In a more general case, the two agents with ongoing effort first receive the cash flow alternately with an increasing frequency of switches and then divide the cash flow at each point of time. This study provides a framework for understanding a broad set of business-contracting issues. The characteristics suggested in the optimal contract help us analyze the causes of business failure such as the recent debacle of mortgage-backed securities.

Key words: optimal contract; incentives; moral hazard in teams; continuous time

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1. Introduction

Businesses often face the problem of providing incentives for agents to work effectively together on projects that develop over time. Many contracting problems involve multiple agents exerting effort at different times. For example, in an entrepreneurial firm, the entrepreneur sets up the business at the outset, whereas the chief executive officer and chief operating officer run its daily operations. In a law firm, partners all exert ongoing effort. In the mortgage banking industry, the loan originator screens borrowers, and the rating agency evaluates the creditworthiness of the mortgage-backed securities (MBS) at the outset, whereas the servicer collects mortgage payments and detects or prevents potential defaults over time. In all of these business collaborations, the division of cash flow over time among the agents affects the agents' effort choices and thus the success of the project.

This paper derives an optimal contract in a continuous-time model in which three agents exert efforts at different times: one at the outset and two over time. For concreteness, the remainder of the paper uses a restaurant parable with an entrepreneur, a chef, and a manager who differ in expertise. The entrepreneur provides effort at the outset to set up the business, whereas the chef and the manager exert ongoing effort to achieve cash flow. The entrepreneur has unique expertise in starting restaurants: selecting

locations, purchasing equipment, designing menus, decorating restaurants, and hiring the chef and manager. The chef has special skills in cooking, and the manager organizes services in the restaurant: hiring people, purchasing supplies, advertising, and managing all other activities that keep the restaurant running smoothly.

These agents' efforts are costly and unobservable, and collectively determine the quality of the restaurant and thus the probability of subsequent business failure. While operating, the restaurant generates cash flow at a fixed rate. Thus, higher effort increases the probability of survival and, consequently, the expected value of cash flow. To focus on incentives, we assume that all agents have linear utility; they are protected by limited liability and are indifferent between receiving a dollar today and receiving it tomorrow. The agents divide all cash flows throughout the life of the restaurant.

We analyze three models: *three-sided moral hazard*, *sequential compensation*, and *proportional sharing*. The *three-sided moral hazard model* captures real-world scenarios with different timing of effort and moral hazard in teams (as described in Holmström 1982). Observe that, for any given amount of payments, the entrepreneur's incentives are not affected by the timing of payments because the up-front effort is sunk once the restaurant is set up, whereas the manager and chef would stop working once the payment flow

stops. Thus, deferring compensation for the manager and chef improves their incentives without impairing the entrepreneur's incentives. In the optimal contract, a predetermined critical date arises, prior to which the entrepreneur obtains all proceeds of the restaurant and after which the manager and chef jointly bear the full variation in cash flow.

The manager and the chef may see one of several compensation patterns after the critical date. When the manager and chef face equally severe moral hazard problems, they share the cash flow equally throughout the project's remaining life. In another case, wherein the severities of moral hazard of the two agents differ, a second critical date arises, prior to which one agent (say, for example, the manager) claims all proceeds and after which the other agent (the chef) becomes the sole claimant of cash flow. In a more general case, the manager and chef split the proceeds first over time (that is, they receive payments alternately), with an increasing frequency of switches. In a finite amount of time, the switches become infinitely frequent, and the system reaches an accumulation point at which the manager and chef start to split the proceeds at each point of time in a fixed proportion that is determined by the relative severities of their moral hazard problems.

To make the basic intuitions more transparent, we decompose the three-sided moral hazard model into two models of two-sided moral hazard. In the *sequential compensation model*, the entrepreneur provides effort at the outset to set up the business, and the manager exerts effort over time, managing as well as cooking. The optimal compensation rule is sequential: All proceeds go to the entrepreneur prior to a predetermined critical date; proceeds go to the manager after the critical date. The lack of payments in early periods serves as a bonding scheme that induces the manager to work hard throughout. Prior to the critical date, the manager must exert effort (though lower than the first-best) to keep the project running so that he may receive payments in the future. After the critical date, the manager becomes the owner of the business and therefore exerts the first-best effort.

The *proportional sharing model* characterizes an ongoing business when the restaurant is fully operational. Over time, both manager and chef expend effort that jointly determines the failure rate of business. The optimal compensation rule splits the proceeds of the restaurant between the two agents in a fixed proportion throughout the life span of the project. The agent facing the more severe moral hazard problem (that is, whose effort is less costly or more critical to the project's survival) receives more proceeds and thus has incentives to exert more effort.

We present several extensions of the three basic models. We also discuss potential efficiency improvement by examining the possibility of a very long-term contract for the agents. This study suggests an approach for understanding a broad set of contracting problems in economics and finance that are far beyond the restaurant example. It rationalizes business convention such as deferred compensation for top executives and profit sharing among business partners. Furthermore, the characteristics suggested in the optimal contract help us analyze the causes of business failure such as the recent mortgage debacle.

1.1. Related Literature

This study contributes to two strands of literature. It extends static multiagent models with dynamics and it complements dynamic single-agent models with moral hazard in teams. Early research on principal-agent or multiple-agent problems focuses on the optimal design of incentive contracts in a static setting. Ross (1973) proposes a linear sharing rule between a (passive) principal and an agent, trading off efficient production and risk sharing. Wilson (1968) works with a setting in which multiple agents all expend effort and shows that the agents share the proceeds proportional to their degrees of risk aversion. Holmström (1982) shows that the first-best outcome cannot be reached in the absence of a budget breaker if the principal also exerts effort.

With a limited-liability constraint, Innes (1990) shows that a "live-or-die" payoff function is optimal in a setting with one agent: the agent receives nothing if the payoff is lower than a certain threshold and claims all of the payoff otherwise. Having much of the same flavor, our sequential compensation model can be interpreted as an extension of Innes' limited-liability model to a multiple-agent case in a continuous-time framework. Innes (1990), however, does not provide much insight into understanding the proportional sharing model and three-sided moral hazard model.

A growing literature on dynamic (mostly continuous-time) agency theory examines intertemporal incentive provisions. In a setting in which the agent's effort affects the drift of the Brownian motion of the output process, Holmström and Milgrom (1987) show that the optimal contract is linear in output if the agent's utility function is exponential. Schättler and Sung (1993) provide a more general mathematical framework, and Sung (1995) and Ou-Yang (2003) show that, in the case of exponential utility, the linearity result holds if the agent's effort affects the diffusion term of the Brownian motion as well.¹ In his

¹ Several recent papers expand the boundary of dynamic agency problems with a single agent. Sannikov (2008) incorporates optimal

seminal paper, Lazear (1979) proposes a long-term implicit labor contract under which young workers are underpaid while old workers are overpaid relative to their marginal contributions. This deferred compensation induces workers to exert effort, because a shirking worker will be fired and the promised wage stream forfeited. Finally, Edmans et al. (2009) provide a dynamic model in which compensation is deferred due to consumption smoothing and manipulation prevention. Notice that all of the dynamic models above include only one agent, whereas this paper addresses dynamic agency problems with multiple agents who exert effort at different times.

In a product warranty context, Dybvig and Lutz (1993) study a two-sided moral hazard problem involving a producer and a consumer. The producer exerts up-front effort that determines the durability of the product, and the consumer exerts effort over time to maintain the product. The effort of the producer and consumer jointly determines the project's failure rate. The consumer pays the producer up front for the ongoing service benefits of the product. Under the optimal contract, the producer pays the consumer (to repair the product) in the event of an early product breakdown.

This paper focuses on a three-sided moral hazard model, with one agent exerting up-front effort and two agents exerting ongoing effort. The introduction of the third agent, who exerts ongoing effort, makes the model applicable to contracting problems involving multiple agents with different timing of effort, such as the entrepreneur, chief executive officer, and chief operating officer in a start-up business. After the agent with up-front effort is fully compensated, the two agents with ongoing effort divide the cash flow under the so-called *singular control*, *bang-bang control*, or *chattering control* for different parameter combinations. This whole set of results cannot be anticipated by reading (Dybvig and Lutz 1993). In addition, this study proposes a solution to a moral hazard problem with multiple agents, all exerting ongoing effort. The proposed *proportional sharing rule* allocates the proceeds at each point of time to the agents in a proportion determined by the relative severities of their moral hazard.

This paper also differs from Dybvig and Lutz (1993) in its solution methodology, assumptions, and economic applications. It makes use of optimal control

theory in constructing and verifying the optimal cash allocation rule, whereas Dybvig and Lutz (1993) rely on a dominance argument, which is of limited use beyond deferring payments for ongoing effort. Moreover, the *pay-as-you-go* system adopted in this study is more useful for real-world scenarios, such as MBS and Real Estate Investment Trust (REIT), in which profits are divided among agents as dividends or coupons at the end of each period due to tax or legal considerations. In contrast, Dybvig and Lutz (1993) endow the consumer with initial funds to purchase the product and transfer wealth from the producer to the consumer in the event of an early product breakdown. Thus, the model in this study is appropriate in a compensation setting, whereas the model in Dybvig and Lutz (1993) is more relevant in a product warranty setting.

The rest of this paper proceeds as follows. Section 2 presents a three-sided moral hazard model with different timing of effort. This three-sided moral hazard model is then decomposed into two models of two-sided moral hazard. Section 3 includes a sequential compensation model with one agent exerting up-front effort and the other agent exerting effort over time; and §4 develops a proportional sharing model with both agents expending ongoing effort. Section 5 provides some examples of potential applications of the model, and §6 concludes. Proofs and detailed derivations are included in the appendix.

2. Three-Sided Moral Hazard

This section provides a model with three-sided moral hazard. Referring to the restaurant example, the entrepreneur exerts effort at the outset when setting up the business, whereas the manager and chef exert effort over time to achieve cash flow. The agents' efforts are costly and unobservable, and jointly affect the probability of subsequent project failure. The cash flow rate is assumed to be constant while the project is running. If a failure occurs, the cash flow becomes zero and remains zero thereafter. We show that, under the optimal contract, the entrepreneur who exerts up-front effort claims all proceeds prior to a known critical date, whereas the manager and chef who exert ongoing effort divide all proceeds afterward. Several patterns are possible for allocating the cash flow between the manager and the chef after the critical date.

The technical setting is as follows. Suppose the initial investment is used to purchase and install equipment specifically geared to the restaurant at the location, such that its resale value upon liquidation is negligible. This sunk initial investment is thus omitted in subsequent analysis. The restaurant has a maximum life span T and generates cash flow at

career paths in his model. DeMarzo and Sannikov (2006) and DeMarzo and Fishman (2007) study dynamic capital structure with privately observable cash flow. They show that the optimal contract resembles a mixture of common equity, long-term debt (or a cash position), and line of credit. DeMarzo et al. (2009) incorporate financing, investment, and uncertainty in demand into a dynamic agency problem. They show that the agent's compensation depends on output price and is deferred when past profits are low.

a fixed rate b while in business. At the outset, the entrepreneur, manager, and chef sign a compensation contract that completely divides the cash flow of the restaurant. While the restaurant is operating, the manager receives $c_1(t)$, the chef receives $c_2(t)$, and the entrepreneur receives $b - c_1(t) - c_2(t)$, where $c_1(t) \geq 0$, $c_2(t) \geq 0$, and $c_1(t) + c_2(t) \leq b$ for $t \in [0, T]$. The assumption above represents financial constraints: All three agents are protected by limited liability, and no one injects cash into the project after the initial investment. On the other hand, they cannot be paid more than the available profits. Essentially, the agents share the cash flow generated by the project over time as dividends or coupons.² This pay-as-you-go setting (without saving of proceeds) approximates real-world scenarios such as amortized mortgage payments and profit sharing among business partners.

Given the incentive contract above, the entrepreneur chooses initial effort $e_0 \in [0, \phi]$ (where ϕ denotes the ideal effort level), whereas the manager and chef choose ongoing effort $e_1(t)$ and $e_2(t)$ (effort intensities: measurable functions from $[0, T]$ to $[0, 1]$). To make the problem analytically tractable while providing useful insight into the dynamic incentive problem with multiple agents, we assume a simple information structure with no intermediate information flow during the project's life. The only uncertainty is the timing of project failure. Our model is more general than single-period models and are simpler than most dynamic models.

Although we can replace the constant cash flow rate in our model by time-dependent cash flow rates, instantaneous cash flow remains independent of effort. The assumption of a fixed cash flow rate captures businesses with relatively stable cash flows, much like food concessions and mortgage portfolios. The moral hazard problem is reflected in the project's failure rate. Alternatively, we could model the influence of effort choices on the magnitude of subsequent cash flow rates. The optimal contract still defers payments to the agents who expend ongoing effort. The case with both effort-dependent cash flows and effort-dependent failure rates adds little economic insight and is not tractable in the current setting.

The formal game has the following time line:

- At the outset, the entrepreneur, the manager, and the chef sign an incentive contract that determines how to share the restaurant's proceeds over time. The three agents then choose their effort plans, and the entrepreneur expends effort up front.

- At any time during the project's life, the manager and chef exert effort and the three agents divide the project's cash flow according to the initial plan.

The more efforts that the agents expend, the higher the project's survival rate and the longer the expected life span. Given a constant cash flow rate, the longer the project survives, the greater the expected cash flow will be. We use a failure rate, rather than the survival rate, merely for modeling convenience. The absolute failure rate (probability density of the failure time) of the project at time s (for $s \in [0, T]$) is

$$f(s; e_0, e_1(\cdot), e_2(\cdot); \phi, m_1, m_2) \\ = (\phi - e_0) + m_1 \int_{\tau=0}^s (1 - e_1(\tau)) d\tau + m_2 \int_{\tau=0}^s (1 - e_2(\tau)) d\tau,$$

where subscripts 0, 1, and 2 represent the entrepreneur, the manager, and the chef, respectively. The entrepreneur's effort choice e_0 has the same influence ($\phi - e_0$) on the failure rate at all future times. The manager and the chef expend effort $e_1(\tau)$ and $e_2(\tau)$ that have impacts $m_1(1 - e_1(\tau))$ and $m_2(1 - e_2(\tau))$, respectively, on the failure rates at all subsequent times. One possible explanation for the cumulative impact is that shirking increases the probability of failure by destroying items related to the productivity of the project. The effects of three sources of moral hazard are additive, and ϕ , m_1 , and m_2 are positive constants.

The probability of failure before time t , $F(t; e_0, e_1(\cdot), e_2(\cdot); \phi, m_1, m_2)$ (simply $F(t)$ hereafter), is given by

$$F(t) = (\phi - e_0)t + m_1 \int_{s=0}^t \int_{\tau=0}^s (1 - e_1(\tau)) d\tau ds \\ + m_2 \int_{s=0}^t \int_{\tau=0}^s (1 - e_2(\tau)) d\tau ds. \quad (1)$$

We require that $\phi T + \frac{1}{2}(m_1 + m_2)T^2 \leq 1$ to ensure that $F(t) \leq 1$ for all feasible effort levels for $t \in [0, T]$.

A hazard rate is an alternative measure to describe the influence of shirking on project failure. The hazard rate, which is conditional on survival until the moment, equals the absolute failure rate divided by the probability of survival. Using the hazard rate usually simplifies the stationarity analysis and makes solution tractable in an infinite-horizon model. With a finite horizon, however, cross terms created by the hazard rate would complicate the algebra. Thus, we use the absolute failure rate in this model.

We assume that all agents have a linear utility of consumption and are indifferent between receiving a dollar today and deferring it until tomorrow.³

²If, alternatively, the entrepreneur deposits an extremely large amount of cash up front, and if the project fails in a very initial stage, the manager and chef seize this deposit. As a result, the manager's and chef's incentives are only distorted in the very initial period. This forcing contract, however, is not feasible in many real-world scenarios.

³The form of the solution is identical if all agents have the same discount rate. Alternatively, the values used in our model can be interpreted as present values, assuming that the cash flow growth rate equals the discount rate.

The cost of effort is strictly convex, differentiable, and increasing, with a zero cost for zero effort so that an interior optimum exists. We choose to use quadratic functions that simplify the algebra while rendering intuitive economic interpretations. Omitting the sunk initial investment, we write the expected social surplus of the project as the expected cash flow net of costs of effort:

$$\Pi = b \int_{t=0}^T (1 - F(t)) dt - \gamma e_0^2 - k_1 \int_{t=0}^T e_1(t)^2 dt - k_2 \int_{t=0}^T e_2(t)^2 dt. \quad (2)$$

At time t , the project’s cash flow is b , whereas the probability that the project reaches t without failure is $1 - F(t)$. Thus, $b \int_{t=0}^T (1 - F(t)) dt$ represents the expected payoff of the project. Terms γ, k_1 , and k_2 (positive constants) represent the unit costs of effort for the entrepreneur, the manager, and the chef, respectively. The cumulative cost of effort for agent i (for $i = 1, 2$) is $k_i \int_{t=0}^T e_i(t)^2 dt$.⁴

Throughout the paper, we choose a cash allocation rule to maximize the expected social surplus subject to the *incentive compatibility* (IC) constraint of each agent.⁵ In solving the maximization problem, we replace the agents’ IC constraints by the first-order conditions. This first-order approach is appropriate because the agents’ maximization problems are convex: The linear utility is separable from the convex disutility of exerting effort. In particular, the entrepreneur chooses effort $e_0 \in [0, \phi]$ to maximize his utility:

$$\Pi_0 = \int_{t=0}^T (b - c_1(t) - c_2(t))(1 - F(t)) dt - \gamma e_0^2,$$

where the first term is the expected payment to the entrepreneur. Substituting for $F(t)$ given in (1), we obtain

$$\Pi_0 = C_1 + e_0 \int_{s=0}^T \int_{\tau=s}^T (b - c_1(\tau) - c_2(\tau)) d\tau ds - \gamma e_0^2, \quad (3)$$

where C_1 collects terms independent of e_0 .⁶

⁴ We assume that the agents incur disutility of exerting effort over the entire horizon of the project. This assumption changes the payoff only to the second order, but makes the impact of effort more separable across time and renders an analytical solution to this dynamic three-sided moral hazard problem. This additional cost of effort can be interpreted as the switching cost to a new job upon a project failure.

⁵ This is one way to find an efficient (second-best) contract, and it has the merit of focusing attention on incentives. Alternatively, we could maximize the utility of one agent subject to the individual rationality constraints of other agents to map out the frontier of efficient contracts.

⁶ $C_1 \equiv \int_{t=0}^T (b - c_1(t) - c_2(t))(1 - \phi t - m_1 \int_{s=0}^t \int_{\tau=0}^s (1 - e_1(\tau)) d\tau ds - m_2 \int_{s=0}^t \int_{\tau=0}^s (1 - e_2(\tau)) d\tau ds) dt$.

Maximizing Π_0 with respect to e_0 (by setting the first-order derivative of (3) at zero) yields the entrepreneur’s optimal effort as

$$e_0 = \frac{1}{2\gamma} \int_{s=0}^T \int_{\tau=s}^T (b - c_1(\tau) - c_2(\tau)) d\tau ds. \quad (4)$$

Point-wise maximizing the manager’s and chef’s utility yields their optimal effort levels as⁷

$$e_1(t) = \frac{m_1}{2k_1} \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds \quad \text{and} \quad (5)$$

$$e_2(t) = \frac{m_2}{2k_2} \int_{s=t}^T \int_{\tau=s}^T c_2(\tau) d\tau ds.$$

We summarize the contracting problem below in Problem 1.

PROBLEM 1. Choose payments to the manager and chef, $c_1(t)$ and $c_2(t)$ (measurable functions from $[0, T]$ to $[0, b]$, satisfying $c_1(t) + c_2(t) \leq b$), to maximize the expected social surplus given in (2) subject to IC constraints given in (4) and (5), where the cumulative failure rate at time t is given in (1).

Changing the order of integrations, we have

$$\int_{s=t}^T \int_{\tau=s}^T c_i(\tau) d\tau ds = \int_{\tau=t}^T c_i(\tau)(\tau - t) d\tau \quad (\text{for } i = 1, 2) \quad \text{and}$$

$$\int_{s=0}^T \int_{\tau=s}^T (b - c_1(\tau) - c_2(\tau)) d\tau ds = \frac{1}{2} b T^2 - \int_{\tau=0}^T (c_1(\tau) + c_2(\tau)) \tau d\tau.$$

Hence, deferring compensation to the manager and chef could increase $\int_{\tau=t}^T (c_1(\tau) + c_2(\tau)) \tau d\tau$ for a time interval with positive measure, while keeping $\int_{\tau=0}^T (c_1(\tau) + c_2(\tau)) \tau d\tau$ unchanged. This improves the manager’s and the chef’s incentives without impairing the entrepreneur’s incentives. The entrepreneur’s effort is sunk at the outset and is thus independent of the payment timing, whereas the manager and chef would stop working if the payment flow stops. The payments in the optimal contract are thus sequential: The entrepreneur receives all proceeds early on, whereas the manager and chef share the proceeds afterward:

$$c_1(\tau) + c_2(\tau) = \begin{cases} 0 & \text{if } \tau < t_d, \\ b & \text{otherwise,} \end{cases}$$

where $t_d \in (0, T)$ is a predetermined critical date, which represents, for example, a major milestone established for managers in entrepreneurial firms.

⁷ See the appendix for detailed derivation.

For notational convenience, we denote the severities of the manager’s and the chef’s moral hazard as $s_1 \equiv m_1^2/4k_1$ and $s_2 \equiv m_2^2/4k_2$, respectively. As m_1 increases, so does the importance of the manager’s effort for the survival of the project. It is then optimal to induce the manager to exert more effort by allocating a greater share of cash flow to him. The size of the share, however, depends also on the costliness of the manager’s effort; that is, a higher k_1 makes it less desirable to induce the manager’s effort.

Assumption 1 summarizes technical conditions.

ASSUMPTION 1. Parameters satisfy

- (i) $\phi T + \frac{1}{2}(m_1 + m_2)T^2 \leq 1$,
- (ii) $(1/4\gamma)bT^2 \leq \phi$,
- (iii) $(m_1/4k_1)bT^2 \leq 1$, and
- (iv) $(m_2/4k_2)bT^2 \leq 1$.

Assumption 1(i) ensures by (1) that $F(t)$ (which is a probability) lies in $[0, 1]$ for all feasible effort levels for $t \in [0, T]$. Assumptions 1(ii), (iii), and (iv) ensure by (4) and (5) that $e_0 \leq \phi$, $e_1(t) \leq 1$, and $e_2(t) \leq 1$ for $t \in [0, T]$.

Whereas we discuss more general cases as model extensions at the end of this section, we now focus on a special case in which the two agents with ongoing effort are equally important; i.e., $s_1 = s_2$. In this case, the manager and chef share the cash flow equally at each point of time throughout the remaining life of the project. Proposition 1 and Lemmas 1 and 2 formally state the results.

PROPOSITION 1. Suppose Assumption 1 holds and that the entrepreneur expends initial effort, whereas the manager and chef exert effort over time. Suppose that the manager and chef face equally severe moral hazard; i.e., $s_1 = s_2$. There exists a unique critical date $t_d \in (0, T)$ given by

$$2t_d^3 - 2Tt_d^2 + 3T^2t_d - \frac{3}{\gamma s_1}(T^2 - t_d^2) = 0, \quad (6)$$

such that (i) the entrepreneur claims all proceeds prior to t_d , namely, $c_1^*(t) = c_2^*(t) = 0$ for $t < t_d$; and (ii) the manager and chef share the proceeds equally after t_d , namely, $c_1^*(t) = c_2^*(t) = \frac{1}{2}b$ for $t \geq t_d$.

PROOF. See the appendix.

Substituting the cash allocation rule described in Proposition 1 into (4) and (5), we have the following lemma:

LEMMA 1. The (second-best) optimal effort choices when $s_1 = s_2$ are

$$e_0^* = \frac{1}{4\gamma}bt_d^2, \quad \text{and for } i = 1, 2,$$

$$e_i^*(t) = \begin{cases} \frac{m_i}{8k_i}b((T-t)^2 - (t_d-t)^2) & \text{if } t < t_d, \\ \frac{m_i}{8k_i}b(T-t)^2 & \text{otherwise.} \end{cases} \quad (7)$$

All three agents exert suboptimal effort. The efficiency loss is increasing as the severity of the agents’ moral hazard increases; more details are provided in the appendix. The budget balance constraint (that is, the three agents completely divide project cash flows) causes the inefficiency in the second-best solution. A budget breaker (Holmström 1982) may help improve efficiency. However, a budget breaker is infeasible if all agents are wealth constrained. Also, a budget breaker may introduce some other problems to the incentive scheme. For example, the budget breaker may collude with one agent to expropriate the others.

The comparative statics summarized below in Lemma 2 are derived from the first-order condition of (6) with respect to $1/\gamma s_1$:

$$\frac{dt_d}{d(1/\gamma s_1)} = \frac{3(T^2 - t_d^2)}{6(t_d - \frac{1}{3}T)^2 + \frac{7}{3}T^2 + 6/\gamma s_1} > 0.$$

LEMMA 2. The critical date t_d (prior to which the entrepreneur claims all proceeds, and after which the manager and chef share all proceeds) increases with the severity of the moral hazard of the entrepreneur relative to that of the manager and chef, $1/\gamma s_1$.

Given $s_1 = s_2$, all statements below about the manager also apply to the chef. Recall that $1/\gamma$ indicates the severity of the entrepreneur’s moral hazard, and s_1 ($\equiv m_1^2/4k_1$) indicates the severity of the manager’s moral hazard. A greater $1/\gamma s_1$ indicates that the entrepreneur’s moral hazard becomes more severe relative to that of the manager. As a result, more proceeds should be allocated to the entrepreneur to improve incentives. With a constant cash flow rate conditional on survival, the entrepreneur receives proceeds over a longer period; that is, t_d is greater.

2.1. Model Extension and Discussion

After the entrepreneur is fully paid, the manager and chef proportionally split the cash flow at each point of time if and only if they have equally severe moral hazard. For some other parameter combinations, a second critical date, $t_{d2} \in (t_d, T)$ arises, prior to which the agent with more severe moral hazard (say, for example, the manager) receives all proceeds, and after which the other agent (in this case, the chef) becomes the sole claimant of the cash flow. This so-called bang-bang control improves the manager’s incentives prior to t_d , and likely prior to t_{d2} even though the manager exerts zero effort after t_{d2} . On the other hand, the chef’s effort is likely to decrease before t_{d2} but is (first-best) optimal after t_{d2} . When the effect of improved incentives dominates that of the worsened incentives, social welfare improves.⁸

⁸ The proof and a numerical example are provided in the electronic companion to this paper, which is available as part of the online version that can be found at <http://mansci.journal.informs.org/>.

In a more general case, the optimal cash allocation between the manager and the chef after t_d is in the form of a chattering control. Under a chattering control, the two agents split the proceeds first over time for a finite amount of time, and then over quantity at each point of time. The pattern of payments can be described as follows. The manager receives all proceeds during $[t_d, t_{d2})$, where $t_{d2} \in (t_d, T)$ is the next predetermined critical date. The chef then claims all cash flow during $[t_{d2}, t_{d3})$, where $t_{d3} \in (t_{d2}, T)$; then the manager receives all proceeds during $[t_{d3}, t_{d4})$, where $t_{d4} \in (t_{d3}, T)$, and so on. The switching of payments accelerates over time and becomes infinitely frequent in a finite amount of time. The system then merges into the *singular arc* on which the manager and chef split the cash flow proportional to the severities of their moral hazard at each point of time for the remaining life of the project.⁹

The first known example of a chattering control is the so-called *Fuller's problem* (Fuller 1963, Wonham 1963). The switches of payments under a chattering control are very similar to the changes of directions of a ball bouncing vertically under the action of gravity. The frequency of payment switches accelerates over time and becomes infinite when reaching the singular arc under the chattering control, while the ball bounces up and down faster and faster before it comes to its final rest.

3. Sequential Compensation

Section 2 analyzes a three-sided moral hazard model with one agent exerting up-front effort and two agents exerting ongoing effort. To make economic intuitions more transparent and to generate additional implications, we decompose the three-sided moral hazard model into a pair of two-sided moral hazard models. Section 4 provides a proportional sharing rule to a model with two agents (in the restaurant example, the manager and chef) exerting ongoing effort. This section includes a sequential compensation model with one agent (the entrepreneur) exerting up-front effort and the other agent (the manager) exerting ongoing effort.

The technical setting is similar to that in §2. At the outset, the two agents sign a compensation contract that allocates $0 \leq c_1(t) \leq b$ to the manager and $b - c_1(t)$ to the entrepreneur for $t \in [0, T]$ while the restaurant is operating. Given this incentive contract,

the entrepreneur chooses initial effort $e_0 \in [0, \phi]$ and the manager chooses ongoing effort $e_1(t)$ (a measurable function from $[0, T]$ to $[0, 1]$). The probability of a failure before t (for $t \in [0, T]$) is determined by the agents' shirking:

$$F(t) = (\phi - e_0)t + m_1 \int_{s=0}^t \int_{\tau=s}^s (1 - e_1(\tau)) d\tau ds. \quad (8)$$

The project's expected social surplus is

$$\Pi = b \int_{t=0}^T (1 - F(t)) dt - \gamma e_0^2 - k_1 \int_{t=0}^T e_1(t)^2 dt. \quad (9)$$

We choose the cash allocation rule to maximize the expected social surplus subject to the IC constraint of each agent. The first-order approach yields the optimal effort of the entrepreneur and manager as

$$\begin{aligned} e_0 &= \frac{1}{2\gamma} \int_{s=0}^T \int_{\tau=s}^T (b - c_1(\tau)) d\tau ds \quad \text{and} \\ e_1(t) &= \frac{m_1}{2k_1} \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds. \end{aligned} \quad (10)$$

We summarize the contract design problem below in Problem 2.

PROBLEM 2. Choose payments to the manager $0 \leq c_1(t) \leq b$ for $t \in [0, T]$ to maximize the expected social surplus given in (9) subject to the IC constraints given in (10), where the cumulative failure rate at time t is given in (8).

Changing the order of integrations, we have

$$\int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds = \int_{\tau=t}^T c_1(\tau)(\tau - t) d\tau.$$

Hence, by (10), deferring payments to the manager increases $\int_{\tau=t}^T c_1(\tau) \tau d\tau$ for a time interval with positive measure while keeping $\int_{\tau=0}^T c_1(\tau) \tau d\tau$ unchanged. This improves the manager's incentives while not impairing the entrepreneur's incentives. As a result, the expected social surplus will be greater under the following sequential allocation rule:

$$c_1(\tau) = \begin{cases} 0 & \text{if } \tau < t_d, \\ b & \text{otherwise,} \end{cases} \quad (11)$$

where $t_d \in (0, T)$ is a predetermined date.

We denote the severity of the moral hazard problem of the manager as $s_1 \equiv m_1^2/4k_1$. Below, Assumption 2 summarizes technical conditions and Proposition 2 describes the optimal compensation contract.

ASSUMPTION 2. Parameters satisfy

- (i) $\phi T + \frac{1}{2}m_1 T^2 \leq 1$,
- (ii) $(1/4\gamma)bT^2 \leq \phi$, and
- (iii) $(m_1/4k_1)bT^2 \leq 1$.

⁹ The cycle durations of switches form a convergent asymptotic geometric progression. Theoretically, it takes an infinite number of switches to reach the singular arc. In practice, the switching points are determined numerically. In many cases, a small number of switches leading to the singular arc can approximate the optimal solution arbitrarily closely; see Chap. 2 of Zelikin and Borisov (1994).

Assumption 2(i) ensures by (8) that $F(t)$ lies in $[0, 1]$ for all feasible effort levels for any $t \in [0, T]$. By (10), Assumptions 2(ii) and (iii) ensure that $e_0 \leq \phi$ and $e_1(t) \leq 1$ for any $t \in [0, T]$.

PROPOSITION 2. *Suppose Assumption 2 holds and that the entrepreneur expends initial effort, whereas the manager exerts effort over time. There exists a unique critical date $t_d \in (0, T)$ given by*

$$t_d^3 + \frac{1}{\gamma s_1} t_d^2 - \frac{1}{\gamma s_1} T^2 = 0, \quad (12)$$

such that (i) the entrepreneur receives all proceeds prior to t_d , namely, $c_1^*(t) = 0$ for $t < t_d$; and (ii) the manager receives all proceeds after t_d , namely, $c_1^*(t) = b$ for $t \geq t_d$.

PROOF. See the appendix.

The optimal effort choices e_0^* and $e_1^*(t)$ can be derived by substituting the solution $c_1(t)$ given in (11) into (10). We have the following lemma:

LEMMA 3. *The (second-best) optimal effort choices of the entrepreneur and the manager are*

$$e_0^* = \frac{1}{4\gamma} b t_d^2 \quad \text{and} \quad e_1^*(t) = \begin{cases} \frac{m_1}{4k_1} b ((T-t)^2 - (t_d-t)^2) & \text{if } t < t_d, \\ \frac{m_1}{4k_1} b (T-t)^2 & \text{otherwise.} \end{cases} \quad (13)$$

Observe that prior to t_d , the manager's optimal effort is below the first-best level and decreases linearly in time. Being the sole owner of the restaurant after t_d , the manager then exerts the first-best effort, which decreases quadratically over time. The entrepreneur, on the other hand, exerts suboptimal effort. The efficiency loss is increasing as the severity of the agents' moral hazard problems increases; see the appendix for details.

The comparative statics are derived from the first-order condition of (12):

LEMMA 4. *The critical date t_d (prior to which the entrepreneur receives all proceeds and after which the manager claims the cash flow) increases with the project life span T and the severity of the entrepreneur's moral hazard relative to the manager's, $1/\gamma s_1$.*

Not surprisingly, as the life span T gets longer, more proceeds (in absolute terms) are allocated to the entrepreneur. In addition, a greater $1/\gamma s_1$ indicates that the entrepreneur is more important for the project's success. Thus, more of the project's proceeds should be allocated to the entrepreneur. Given a constant cash flow rate, the entrepreneur receives cash flows over a longer period; i.e., the critical date t_d is greater.

3.1. Model Extension and Discussion

The contract with sequential compensation may seem to be too restrictive because we assume a pay-as-you-go mechanism and the cash flow allocation does not explicitly depend on the timing of project failure. It is actually more general than it appears to be. Consider an alternative institutional setting wherein a very long-term contract is permitted, similar to the one used in Lazear (1979). Suppose all proceeds of the project are deposited in an escrow account and are distributed to the agents upon a project failure or at T if the project reaches its life span. The concept of the optimal incentive scheme in this alternative setting is the same as that stated in Proposition 2: the entrepreneur who exerts up-front effort bears the cost of an early project failure.

The incentive contract is characterized by the timing of project failure. There exists a known critical date t^* . If the project fails before t^* , the manager claims all proceeds in the escrow account. This allocation rule penalizes the entrepreneur for an early failure that is more likely due to his shirking at the outset. On the other hand, if the project fails after t^* , the entrepreneur claims all proceeds in the escrow account. This penalizes the manager for not working hard because a failure in later periods is more likely due to the manager's shirking over time. The critical date t^* is decreasing as the manager's moral hazard becomes more severe relative to the entrepreneur's. This improves the incentives of the more important agent to work hard. Finally, if the project survives until its life span T , the two agents divide the proceeds in the escrow account. The final contingent payments serve as bonuses to provide incentives for both agents, particularly, to induce the manager to expend effort after t^* . Reaching the project's life span can be interpreted as the IPO or buyout of the entrepreneurial firm.¹⁰

Without saving costs, forced saving improves social welfare.¹¹ However, saving is wasting assets if the discount rate is much higher than the interest rate in practice. In addition, forced saving is not practical in many real-world scenarios such as MBS and REIT due to legal or tax considerations. Moreover, it may not be feasible to save all proceeds and make the agents continue borrowing against their future income.¹²

¹⁰ The compensation schedule with an escrow account works as if inferences about agents' effort choices are made based on the time of project failure, even though agents' effort levels have been determined at the outset (similar to the use of the likelihood function in compensating the agent in Holmström 1979).

¹¹ The proof and a numerical example are provided in the electronic companion to this paper.

¹² This institutional setting suggests a situation similar to the indentured servitude in the American South in the 17th and 18th centuries.

4. Proportional Sharing

To gain additional insight into the dynamic moral hazard in team problem, this section includes a two-sided moral hazard model in which the manager and chef exert efforts over time that jointly affect the survival of an ongoing business. At the outset, the two agents sign a compensation contract that allocates $0 \leq c_1(t) \leq b$ to the manager and $b - c_1(t)$ to the chef for $t \in [0, T]$ while the restaurant is operating. Under this contract, the manager and chef choose effort plans, $e_1(t)$ and $e_2(t)$ (measurable and normalized to fall in $[0, 1]$ for $t \in [0, T]$). We show below that under the optimal contract, the two agents share the cash flow at each point of time in a proportion determined by the severities of their moral hazard problems.

The probability that the project fails before t (for $t \in [0, T]$) is given by

$$F(t) = m_1 \int_{s=0}^t \int_{\tau=0}^s (1 - e_1(\tau)) d\tau ds + m_2 \int_{s=0}^t \int_{\tau=0}^s (1 - e_2(\tau)) d\tau ds. \tag{14}$$

The expected social surplus is

$$\Pi = b \int_{t=0}^T (1 - F(t)) dt - k_1 \int_{t=0}^T e_1(t)^2 dt - k_2 \int_{t=0}^T e_2(t)^2 dt. \tag{15}$$

Each agent’s strategy satisfies the IC constraint. The first-order approach yields the optimal effort levels of the manager and chef as

$$e_1(t) = \frac{m_1}{2k_1} \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds \quad \text{and} \tag{16}$$

$$e_2(t) = \frac{m_2}{2k_2} \int_{s=t}^T \int_{\tau=s}^T (b - c_1(\tau)) d\tau ds.$$

We summarize the contract design problem below in Problem 3.

PROBLEM 3. Choose payments to the manager $0 \leq c_1(t) \leq b$ for $t \in [0, T]$ to maximize the expected social surplus given in (15) subject to the IC constraints given in (16), where the cumulative failure rate at time t is given in (14).

Substituting IC constraints given in (16) into (15), changing the order of integrations, and rearranging terms, we obtain the expected social surplus as

$$\Pi = C_3 - (s_1 + s_2) \cdot \int_{t=0}^T \left(\int_{s=t}^T \int_{\tau=s}^T \left(c_1(\tau) - \frac{s_1}{s_1 + s_2} b \right) d\tau ds \right)^2 dt, \tag{17}$$

where C_3 collects terms independent of $c_1(\tau)$ and $s_i \equiv m_i^2/4k_i$ represents the severity of moral hazard of

agent i (for $i = 1, 2$). Given that the second term of (17) is nonnegative, the maximum of Π is reached when the second term equals zero. Namely, the objective function (17) is maximized point-wise (for $t \in [0, T]$) when

$$\int_{s=t}^T \int_{\tau=s}^T \left(c_1(\tau) - \frac{s_1}{s_1 + s_2} b \right) d\tau ds = 0.$$

Therefore, the payments to the manager and chef are (for $t \in [0, T]$)

$$c_1^*(t) = \frac{s_1}{s_1 + s_2} b \quad \text{and} \quad b - c_1^*(t) = \frac{s_2}{s_1 + s_2} b.$$

We summarize technical conditions in Assumption 3 and the main result in Proposition 3.

ASSUMPTION 3. Parameters satisfy

- (i) $\frac{1}{2}(m_1 + m_2)T^2 \leq 1$,
- (ii) $(m_1/4k_1)(s_1/(s_1 + s_2))bT^2 \leq 1$, and
- (iii) $(m_2/4k_2)(s_2/(s_1 + s_2))bT^2 \leq 1$.

Assumption 3(i) ensures by (14) that $F(t)$ lies in $[0, 1]$ for all feasible effort levels for any $t \in [0, T]$. Assumptions 3(ii) and (iii) ensure by (16) that $0 \leq e_1(t) \leq 1$ and $0 \leq e_2(t) \leq 1$ for any $t \in [0, T]$.

PROPOSITION 3. Suppose Assumption 3 holds and that both manager and chef exert ongoing effort. The two agents share the project’s proceeds in a fixed proportion: $c_1^*(t) = (s_1/(s_1 + s_2))b$ and $b - c_1^*(t) = (s_2/(s_1 + s_2))b$, where $s_1 \equiv m_1^2/4k_1$ and $s_2 \equiv m_2^2/4k_2$ denote the severities of the moral hazard of the manager and chef, respectively.

PROOF. See the derivation above.¹³

The compensation contract resembles equity-like claims.¹⁴ At each point in time, the manager and chef split the cash flow, with the more important agent (a higher s_i) receiving a larger fraction. Economically, it is the quadratic cost of effort for each agent that prevents us from pushing one agent to work too hard while keeping the other agent idling anytime during the project’s life. We summarize the optimal effort choices as follows.

¹³ An alternative proof of Proposition 3 using the maximum principle is available upon request. Essentially, the costate variable (switching function) is identically zero under the proportional sharing rule and we have a singular control. The fourth-order derivative of the switching function yields $c_1^*(t) = (s_1/(s_1 + s_2))b$ for $t \in [0, T]$.

¹⁴ Casamatta (2003) and Admati and Pfleiderer (1994) model a two-sided moral hazard problem in a static setting in the context of VC contracting. Chisholm (1997) and Weinstein (1998) examine empirically whether agency problems are the driving force for using profit share contract in the motion picture industry. Bhattacharya and Lafontaine (1995) summarize the theory and evidence on two-sided moral hazard problems.

LEMMA 5. *The (second-best) optimal effort choices of the manager and chef (for $t \in [0, T]$) are*

$$\begin{aligned} e_1^*(t) &= \frac{m_1}{4k_1} \frac{s_1}{s_1 + s_2} b(T-t)^2 \quad \text{and} \\ e_2^*(t) &= \frac{m_2}{4k_2} \frac{s_2}{s_1 + s_2} b(T-t)^2. \end{aligned} \quad (18)$$

Observe that the optimal effort decreases over time. The agent whose effort is more critical to the survival of the project receives greater cash flow and thus shirks less relative to the first-best effort. The efficiency loss relative to the first-best is increasing as the severities of the agents' moral hazard increase; details are provided in the appendix.

4.1. Model Extension and Discussion

The proportional sharing rule is optimal in more general settings. Suppose more than two agents exert ongoing effort that jointly affects subsequent project failure. It can be shown that under the optimal contract, the agents divide the proceeds at each point of time in a fixed proportion determined by the relative severities of moral hazard. The optimal sharing rule in a case with three agents is provided in the appendix.

In addition, the proportional sharing rule applies when the cash flow rates and costs of effort vary over time. Under the optimal contract, agents share time-variant proceeds in proportions determined by the time-variant severities of their moral hazard (for $t \in [0, T]$):

$$c_1^*(t) = \frac{s_1(t)}{s_1(t) + s_2(t)} b(t) \quad \text{and} \quad c_2^*(t) = \frac{s_2(t)}{s_1(t) + s_2(t)} b(t),$$

where $s_1(t) \equiv m_1^2/4k_1(t)$ and $s_2(t) \equiv m_2^2/4k_2(t)$.

One may wonder why the bang-bang control (that is, paying the manager and chef sequentially), as discussed in the extension of the three-sided moral hazard model after the entrepreneur receives all payments ($t \geq t_d$), is not optimal when only the two agents with ongoing effort exist throughout the project's life span. One may also wonder why the manager and chef do not share the proceeds in a fixed proportion after t_d in the three-sided moral hazard model. Actually they do, but only in a special case when they have the same severity of moral hazard. The answers to these two questions are the same.

Let us consider the three-sided moral hazard model. Mathematically, the existence of the entrepreneur and the time interval in which the entrepreneur claims all proceeds, $t < t_d$, change the maximization problem for the manager and chef after t_d . It is no longer feasible (except in a knife-edge case wherein the two agents have the same severity of moral hazard) to achieve the maximum by splitting the proceeds between the

manager and the chef in a fixed proportion throughout the remaining life of the project. More specifically, maximizing the expected social surplus is a trade-off between maximizing terms integrating over $[t_d, T]$ and maximizing a term integrating over the project's life span, $-1/4\gamma(\int_{s=0}^T \int_{\tau=s}^T (c_1(\tau) + c_2(\tau)) d\tau ds)^2$; see (20) and (21) in the appendix. Splitting the proceeds proportionally maximizes the integrals over $[t_d, T]$, but it is more than offset by the value change of the integral over $[0, T]$ in all but the knife-edge case. In contrast, when only the two agents with ongoing effort exist throughout the project's life span, the integral over $[0, T]$ is a constant because $c_1(t) + c_2(t) = b$ for any $t \in [0, T]$, whereas the integrals over $[t_d, T]$ are maximized when the two agents split the proceeds proportionally at each point of time; see (17).

Economically, by (4) and (5), the cash allocation between the manager and the chef after t_d affects the effort choices of the agents in all earlier periods, including periods prior to t_d . These effort choices, by (1), affect the failure rates in all subsequent periods, including periods before and after t_d . These failure rates then interact with payments to the agents in determining the expected social surplus. This *feedback* feature makes the early periods in which the entrepreneur plays an active role undetachable from the later periods in which the manager and chef share all the cash flow. As a result, sharing the cash flow proportionally between the manager and chef at each point of time after t_d is not optimal in general.

5. Examples of Potential Applications

This study provides a framework for understanding a broad set of contracting issues in practice far beyond the restaurant context. In this section, we discuss some potential applications of the sequential compensation model and proportional sharing model. First, let us look at the recent debacle of mortgage-backed securities. Generated through a process known as *Securitization*, MBS are debt obligations that represent claims to the cash flows from pools of mortgage loans. A typical securitization process can be described as follows. A loan originator makes loans to borrowers, then sells these loans to an MBS issuer. Under the advice of a rating agency, the issuer (often via a special purpose vehicle) pools the loans and repackages claims to their future cash flows into securities. At the same time, a servicer is established to collect and process future mortgage payments. The rated securities are then sold to investors.

In the discussion below, we focus on the roles of the loan originator, the rating agency, and the servicer. The loan originator and rating agency mainly exert effort up front to ensure the quality of the underlying

loans and the securities they back, whereas the servicer works over time to ensure timely collections of mortgage payments from borrowers.

The role of the servicer, which includes monitoring the quality of underlying property, restructuring payment streams, and foreclosing property when needed, is critical to improve cash flows, especially for securities backed by subprime loans. In the current system, a servicer typically receives 25–50 basis points of the unpaid principal balance as compensation for providing such services. These fees are front-end loaded because the unpaid principal balance decreases over time. Our model suggests that the opposite is true: deferring payments provides stronger incentives for the servicer to execute best practices to improve loan performance. If we take the model literally, the servicer should receive no payment in early years and claim all proceeds in later years of the securities' term.

Currently, the loan originator receives all payments at the outset from the issuer. Although a recourse clause on early payment default is included in some loan sales, it typically covers a very short period, mainly to protect the issuer from fraud rather than to motivate the originator to ensure loan quality. Our model suggests that payments to the originator should come ahead of payments to the servicer. More importantly, payments to the originator should be linked to subsequent loan performance. One way to achieve this is to pay the loan originator over time out of residual cash flows so that the originator bears some of the default risk.¹⁵ On the other hand, if the originator insists on receiving a lump-sum payment at the outset, the originator should offer recourse that covers a sufficiently long period to allow payment default to occur.

The rating agency analyzes the pool of loans and advises the issuer on how to structure the loans and enhance the credit of MBS (such as geographic diversification) to achieve a specific rating. Paid by the issuer and pressed by competitors, the rating agency conceivably has strong incentives to help the issuer maximize the size of securities with higher ratings for any given pool of loans. When the reputation effect alone is not sufficient to solve the problem of loosened rating standards, our model suggests linking the payments to the rating agency to subsequent performance of the loans that back the rated securities. For example, giving rated securities as payment for initial rating fees and making the rating agency hold those securities long enough (so the rating agency faces some default risk) will eliminate

the rating agency's incentives to inflate ratings.¹⁶ On the other hand, competition for business will deter the rating agency from unfairly depressing ratings. We acknowledge that factors beyond the agents' control (such as exceptionally adverse changes in home prices and interest rates) also affect payment default, which may temper the incentives provided to the agents in the optimal contract. Although our model cannot be prescriptive, it provides a framework for detailed discussion and analysis of business practices.

This study also helps us understand the wisdom of business convention. The proportional sharing model applies to contracting problems involving multiple agents who exert effort together over time. Even though some interesting features such as the existence of paralegal are omitted, the propositional sharing model still reflects the spirit of profit sharing among important players at law firms. Before making partners, associates work long hours and receive very little compensation. After promotion, junior partners and senior partners split the profits proportionally to their seniorities. This mechanism works in practice because seniority is considered an appropriate proxy for a partner's importance and contribution to the business. In particular, a senior partner in general has the reputation and ability to procure business, improve efficiency, and mentor associates as they progress in their careers.

The proportional sharing model also applies to contracting between the distributor (sometimes the studio) and exhibitor (movie theaters) of a motion picture. The studio/distributor exerts effort to promote the movie mainly during early weeks, whereas the exhibitor provides in-house advertisement and usher services. They typically split the box office revenue, with the distributor's share starting at 70% in the opening week and decreasing over the weeks to come. This adjustment of splits can be attributed to the decreasing influence of the distributor's marketing strategy on gross receipts.

6. Conclusion

This paper examines a dynamic contracting problem with three-sided moral hazard. One agent exerts effort at the outset to set up the business and two agents exert ongoing effort to achieve cash flows. The agents' efforts jointly affect the probability of subsequent project failure and thus its expected cash flow. The timing of payments should reflect the timing of effort. We show that under the optimal incentive contract, the agent with up-front effort is penalized for an early project failure that is more likely due to his

¹⁵ Our model leaves out financing. Payments to the agents should have lower priorities than payments to investors to make the agents bear default risk.

¹⁶ The rating agency should unload the securities before issuing updated ratings to prevent rating biases.

shirking at the outset, whereas the agents with ongoing effort are penalized for a failure occurring in later periods. In particular, a predetermined critical date arises, prior to which the agent with up-front effort claims all cash flows, and after which the two agents with ongoing effort divide the project's proceeds. Several patterns are possible for allocating cash flows between the two agents with ongoing effort after the critical date. In a special case wherein their moral hazard problems are equally severe, the two agents split the proceeds equally at each point of time. The model provides a framework for detailed analysis of dynamic profit sharing among business partners in law firms, the movie industry, and mortgage banking industry.

Our research suggests some directions for future research on dynamic contracting with multiple agents. For example, our model assumes that each agent exerts only one type of effort, either up-front or ongoing. Suppose, alternatively, that there are two agents with three roles. Using the restaurant example, the entrepreneur/manager not only exerts effort at the outset to set up the business but also works over time to manage its daily operations. Meanwhile, the chef exerts ongoing effort to ensure food quality. The general model remains unsolved. In one special case wherein the dual effort is nontrivial relative to the chef's effort, we show that the entrepreneur/manager collects all proceeds prior to a known critical date, whereas the chef receives all proceeds afterward.

Finally, in our model, the agents' efforts have additive effects on subsequent project failure. In some real-world applications, however, if one agent (say, the chef in the restaurant example) exerts zero effort, the business is destined to fail regardless of what effort other agents exert. We leave it to future research to model the complementarity of agents' efforts in the production function.

7. Electronic Companion

An electronic companion to this paper is available as part of the online version that can be found at <http://mansci.journal.informs.org/>.

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Appendix

Derivation of (5)

The manager's utility is given by

$$\begin{aligned} \Pi_1 &= \int_{t=0}^T c_1(t)(1 - F(t)) dt - k_1 \int_{t=0}^T e_1(t)^2 dt \\ &= C_2 + m_1 \int_{t=0}^T e_1(t) \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds dt - k_1 \int_{t=0}^T e_1(t)^2 dt, \end{aligned}$$

where C_2 collects terms independent of $e_1(t)$. Maximizing Π_1 point-wise with respect to $e_1(t)$ yields the manager's optimal effort as

$$e_1(t) = \frac{m_1}{2k_1} \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds.$$

Similarly, we obtain the chef's optimal effort as

$$e_2(t) = \frac{m_2}{2k_2} \int_{s=t}^T \int_{\tau=s}^T c_2(\tau) d\tau ds. \quad \square$$

The proof of Proposition 1 applies the results of Lemmas 6–8, which are stated below the proof.

Proof of Proposition 1

Substituting the IC constraints given in (4) and (5) into (1) and (2) and omitting the terms independent of $c_1(t)$ and $c_2(t)$, we obtain the following expected social surplus:¹⁷

$$\begin{aligned} \hat{\Pi} &= -\frac{1}{4\gamma} (x_1(0) + y_1(0))^2 + s_1 b \int_{t=0}^T x_1(t)(T - t)^2 dt \\ &\quad - s_1 \int_{t=0}^T x_1(t)^2 dt + s_2 b \int_{t=0}^T y_1(t)(T - t)^2 dt \\ &\quad - s_2 \int_{t=0}^T y_1(t)^2 dt, \end{aligned} \quad (19)$$

where

$$x_1(t) \equiv \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds \quad \text{and} \quad y_1(t) \equiv \int_{s=t}^T \int_{\tau=s}^T c_2(\tau) d\tau ds.$$

Define state variables $x_2(t) = \dot{x}_1(t)$ and $y_2(t) = \dot{y}_1(t)$; then we have $\dot{x}_2(t) = c_1(t)$ and $\dot{y}_2(t) = c_2(t)$. The optimization problem is transformed to the following:

PROBLEM 1M. Choose $(c_1(t), c_2(t))$ for $t \in [0, T]$ from the admissible set of controls

$$c_1(t) \geq 0, \quad c_2(t) \geq 0, \quad \text{and} \quad c_1(t) + c_2(t) \leq b$$

¹⁷The expected social surplus is $\Pi = \hat{\Pi} + (bT - \frac{1}{2}\phi bT^2 - \frac{1}{6}(m_1 + m_2)bT^3 + (1/16\gamma)b^2T^4)$. In the remainder of the proof, $\hat{\Pi}$ is referred to as the expected social surplus.

to maximize $\widehat{\Pi}$ given in (19) subject to system dynamics

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), & \dot{x}_2(t) &= c_1(t), \\ \dot{y}_1(t) &= y_2(t), & \dot{y}_2(t) &= c_2(t), \end{aligned}$$

and terminal conditions

$$\begin{aligned} x_1(T) &= 0, & x_2(T) &= 0, \\ y_1(T) &= 0, & y_2(T) &= 0. \end{aligned}$$

The Hamiltonian is

$$H = s_1 b x_1(t)(T-t)^2 - s_1 x_1(t)^2 + s_2 b y_1(t)(T-t)^2 - s_2 y_1(t)^2 + \lambda_1(t)x_2(t) + \lambda_2(t)c_1(t) + \mu_1(t)y_2(t) + \mu_2(t)c_2(t), \quad (20)$$

and the initial value is

$$\phi(x_1(0), y_1(0)) = -\frac{1}{4\gamma}(x_1(0) + y_1(0))^2. \quad (21)$$

In the Hamiltonian above, $x_1(t)$, $x_2(t)$, $y_1(t)$, and $y_2(t)$ are state variables; $\lambda_1(t)$, $\lambda_2(t)$, $\mu_1(t)$, and $\mu_2(t)$ are costate variables ($\lambda_2(t)$ and $\mu_2(t)$ are also called *switching functions*); and $c_1(t)$ and $c_2(t)$ are the control variables to be determined.

Notice that the Hamiltonian is linear in control variables $c_1(t)$ and $c_2(t)$. The building blocks of the optimal control can only be a bang-bang control (at any instant, a single agent claims all proceeds) or a singular control (the manager and chef split the cash flow in a fixed proportion).¹⁸ Specifically, we have

$$\begin{aligned} c_1(t) &= 0, & c_2(t) &= 0 & \text{if } \lambda_2(t) < 0, \mu_2(t) < 0; \\ c_1(t) &= b, & c_2(t) &= 0 & \text{if } \lambda_2(t) \geq 0, \lambda_2(t) > \mu_2(t); \\ c_1(t) &= 0, & c_2(t) &= b & \text{if } \mu_2(t) \geq 0, \mu_2(t) > \lambda_2(t); \text{ and} \\ c_1(t) &= \frac{s_1}{s_1 + s_2}b, & c_2(t) &= \frac{s_2}{s_1 + s_2}b & \text{if } \lambda_2(t) = \mu_2(t) \geq 0. \end{aligned}$$

We determine switching functions $\lambda_2(t)$ and $\mu_2(t)$ using the following dynamics:

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1(t)} = -s_1 b(T-t)^2 + 2s_1 x_1(t), \\ \lambda_1(0) &= -\frac{\partial \phi(x_1(0), y_1(0))}{\partial x_1(0)} = \frac{1}{2\gamma}(x_1(0) + y_1(0)), \end{aligned} \quad (22)$$

$$\dot{\lambda}_2(t) = -\frac{\partial H}{\partial x_2(t)} = -\lambda_1(t),$$

$$\lambda_2(0) = 0;$$

and

$$\begin{aligned} \dot{\mu}_1(t) &= -\frac{\partial H}{\partial y_1(t)} = -s_2 b(T-t)^2 + 2s_2 y_1(t), \\ \mu_1(0) &= -\frac{\partial \phi(x_1(0), y_1(0))}{\partial y_1(0)} = \frac{1}{2\gamma}(x_1(0) + y_1(0)), \end{aligned} \quad (23)$$

$$\dot{\mu}_2(t) = -\frac{\partial H}{\partial y_2(t)} = -\mu_1(t),$$

$$\mu_2(0) = 0.$$

¹⁸ CLAIM. If $\lambda_2(t) = \mu_2(t) \geq 0$, we have $c_1(t) = (s_1/(s_1 + s_2))b$ and $c_2(t) = (s_2/(s_1 + s_2))b$.

PROOF. If $\lambda_2(t) = \mu_2(t) \geq 0$, we have $c_1(t) + c_2(t) = b$. Define $\psi(t) \equiv \lambda_2(t) - \mu_2(t) = 0$; then all the derivatives of $\psi(t)$ equal zero. In particular, $\dot{\psi}(t) = 0$ yields $c_1(t) = (s_1/(s_1 + s_2))b$. This concludes the proof.

By (22) and (23), the switching functions have initial values of zero: $\lambda_2(0) = \mu_2(0) = 0$. Lemma 6 below shows that $\lambda_2(t)$ and $\mu_2(t)$ dip first ($\dot{\lambda}_2(0) < 0$ and $\dot{\mu}_2(0) < 0$). They then increase and cross zero from below (Lemmas 7 and 8). The two switching functions cross zero at the same moment if and only if $s_1 = s_2$, which is the case discussed in Proposition 1. Define $t_d = \inf\{t > 0, \lambda_2(t) = \mu_2(t) \geq 0\}$. The crossing point t_d is shown to be unique. Additionally, $\lambda_2(t) = \mu_2(t)$ remains positive after t_d given the convexity of $\lambda_2(t)$ and $\mu_2(t)$. Hence, the cash allocation rule specified in Proposition 1 is optimal. \square

Because $\lambda_2(t) = \mu_2(t)$ for any $t \in [0, T]$ when $s_1 = s_2$, we focus on conditions for $\lambda_2(t)$ in the remainder of the proof.

LEMMA 6. $\dot{\lambda}_2(0) < 0$.

PROOF. By the definitions of $x_1(0)$ and $y_1(0)$, we have

$$\dot{\lambda}_2(0) = -\lambda_1(0) = -\frac{1}{2\gamma}(x_1(0) + y_1(0)) \leq 0.$$

We prove by contradiction that $\dot{\lambda}_2(0) < 0$. Suppose not. Then $x_1(0) + y_1(0) = 0$ implies that $c_1(t) = c_2(t) = 0$ for any $t \in (0, T]$. Thus, $x_1(t) = y_1(t) \equiv 0$, which implies that $\ddot{\lambda}_2(t) = -\dot{\lambda}_1(t) = s_1 b(T-t)^2 > 0$. Given that $\lambda_2(0) = 0$ and $\dot{\lambda}_2(0) = 0$, we have $\lambda_2(t) > 0$ for any $t \in (0, T]$. But recall that $\lambda_2(t) \equiv \mu_2(t)$, we have $c_1(t) = c_2(t) = \frac{1}{2}b$ for any $t \in (0, T]$; a contradiction! \square

LEMMA 7. $\ddot{\lambda}_2(t) > 0$ for any $t \in [0, T]$.

PROOF. By the definition of $x_1(t)$, we have

$$\ddot{\lambda}_2(t) = -\dot{\lambda}_1(t) = s_1 b(T-t)^2 - 2s_1 x_1(t) \geq 0.$$

We prove by contradiction that $\ddot{\lambda}_2(t) > 0$. Suppose not. Then there exists $t \in [0, T]$ such that $\dot{\lambda}_2(t) = \ddot{\mu}_2(t) = 0$, which implies that $x_1(t) = y_1(t) = \frac{1}{2}b(T-t)^2$. This contradicts the condition of $x_1(t) + y_1(t) \leq \frac{1}{2}b(T-t)^2$. \square

LEMMA 8. There exists a unique $t_d \in (0, T)$ such that the switching function crosses zero at t_d ; i.e., $\lambda_2(t_d) = 0$, and stays positive afterward; i.e., $\lambda_2(t) > 0$ for $t > t_d$.

PROOF. By Lemmas 6 and 7, $\lambda_2(t)$ starts at zero, dips first, and crosses zero from below at $t_d > 0$; i.e., $\lambda_2(t_d) = 0$. We first prove by contradiction that $\dot{\lambda}_2(t_d) > 0$, then show the existence of a unique $t_d \in (0, T)$. Suppose $\dot{\lambda}_2(t_d) \leq 0$. Given that $\dot{\lambda}_2(0) < 0$, and $\dot{\lambda}_2(t) > 0$ for any $t \in [0, T]$, we have $\lambda_2(t) < 0$ for $t \in (0, t_d)$. Given $\lambda_2(0) = 0$, we have $\lambda_2(t_d) < 0$; a contradiction!

Given that $\lambda_2(t_d) = 0$, $\dot{\lambda}_2(t_d) > 0$, and $\ddot{\lambda}_2(t) > 0$ for any $t \in [0, T]$, we have $\lambda_2(t) > 0$ for $t > t_d$. When $s_1 = s_2$, $\lambda_2(t) = \mu_2(t)$ for any $t \in [0, T]$. Thus, the control variables are

$$\begin{cases} c_1^*(t) = c_2^*(t) = 0 & \text{if } t < t_d, \\ c_1^*(t) = c_2^*(t) = \frac{1}{2}b & \text{otherwise.} \end{cases}$$

The state variable $x_1(t)$ is

$$x_1(t) = \frac{1}{2} \frac{s_1}{s_1 + s_2} b((T-t)^2 - (t_d - t)^2) \quad \text{if } t \leq t_d. \quad (24)$$

Substituting (24) into (22), we have the dynamics and initial condition of the costate variable $\lambda_1(t)$ as

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{s_1 s_2}{s_1 + s_2} b(T-t)^2 - \frac{s_1^2}{s_1 + s_2} b(t_d - t)^2 \quad \text{if } t \leq t_d, \quad \text{and} \\ \lambda_1(0) &= \frac{1}{2\gamma} (x_1(0) + y_1(0)) = \frac{1}{4\gamma} b(T^2 - t_d^2). \end{aligned} \quad (25)$$

Substituting (25) into (22), we rewrite the switching function $\lambda_2(t)$ for $t \leq t_d$ as

$$\begin{aligned} \lambda_2(t) &= -\frac{1}{4\gamma} b(T^2 - t_d^2)t + \frac{1}{12} \frac{s_1 s_2}{s_1 + s_2} b(6T^2 t^2 - 4Tt^3 + t^4) \\ &\quad + \frac{1}{12} \frac{s_1^2}{s_1 + s_2} b(6t_d^2 t^2 - 4t_d t^3 + t^4). \end{aligned}$$

The condition that $\lambda_2(t_d) = 0$ yields either $t_d = 0$ or

$$Q(t_d) \equiv 2t_d^3 - 2Tt_d^2 + 3T^2 t_d - \frac{3}{\gamma s_1} (T^2 - t_d^2) = 0. \quad (26)$$

Because

$$\begin{aligned} Q(t_d | t_d = 0) &= -\frac{3}{\gamma s_1} T^2 < 0, \\ Q(t_d | t_d = T) &= 3T^3 > 0, \quad \text{and} \\ Q'(t_d) &= 6(t_d - \frac{1}{3}T)^2 + \frac{7}{3}T^2 + \frac{6}{\gamma s_1} t_d > 0, \end{aligned}$$

there exists a unique solution $t_d \in (0, T)$ to (26). \square

The proof of Proposition 2 is similar to the proof of Proposition 1. It applies the results of Lemmas 9–11, which are stated below the proof.

Proof of Proposition 2

For notational convenience, define $x_1(t) \equiv \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds$. Substituting the IC constraints of the entrepreneur and the manager (10) into the objective function (9) and omitting the terms independent of $c_1(t)$, we have¹⁹

$$\begin{aligned} \bar{\Pi} &= -\frac{1}{4\gamma} x_1(0)^2 + s_1 b \int_{t=0}^T x_1(t)(T-t)^2 dt \\ &\quad - s_1 \int_{t=0}^T x_1(t)^2 dt. \end{aligned} \quad (27)$$

Define $x_2(t) = \dot{x}_1(t)$; then $\dot{x}_2(t) = c_1(t)$. The optimization problem is transformed to:

PROBLEM 2M. Choose $0 \leq c_1(t) \leq b$ for $t \in [0, T]$ to maximize $\bar{\Pi}$ given in (27) subject to system dynamics

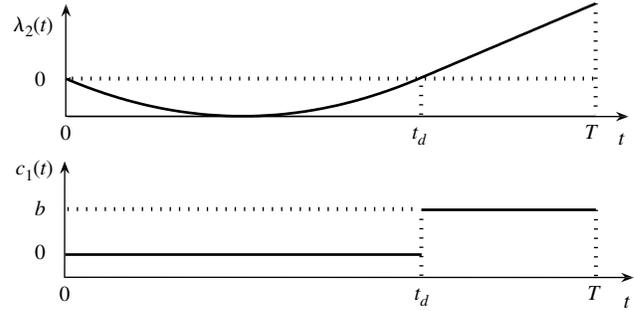
$$\dot{x}_1(t) = x_2(t), \quad \dot{x}_2(t) = c_1(t),$$

and terminal conditions

$$x_1(T) = 0, \quad x_2(T) = 0.$$

¹⁹ Now the expected social surplus is $\Pi = \bar{\Pi} + (bT - \frac{1}{2}\phi bT^2 - \frac{1}{6}m_1 bT^3 + (1/16\gamma)b^2 T^4)$. In the remainder of the proof, $\bar{\Pi}$ is referred to as the expected social surplus.

Figure A.1 The Switching Function $\lambda_2(t)$ and the Control $c_1(t)$ (Payments to the Manager) Over the Project's Life Span



Notes. The switching function $\lambda_2(t)$ is convex. It starts from zero, dips first, crosses zero from below at the critical date $t_d \in (0, T)$, and remains positive after t_d . Correspondingly, $c_1(t)$ is zero (the minimum payment) prior to t_d and is b (the maximum payment) afterward.

The Hamiltonian is

$$H = s_1 b x_1(t)(T-t)^2 - s_1 x_1(t)^2 + \lambda_1(t)x_2(t) + \lambda_2(t)c_1(t),$$

and the initial value is

$$\phi(x_1(0)) = -\frac{1}{4\gamma} x_1(0)^2.$$

Because the Hamiltonian is linear in the control variable $c_1(t)$, we have²⁰

$$c_1(t) = \begin{cases} 0 & \text{if } \lambda_2(t) < 0, \\ b & \text{if } \lambda_2(t) \geq 0. \end{cases}$$

The switching function $\lambda_2(t)$ is determined by the following dynamics:

$$\begin{aligned} \dot{\lambda}_1(t) &= -\frac{\partial H}{\partial x_1(t)} = -s_1 b(T-t)^2 + 2s_1 x_1(t), \\ \lambda_1(0) &= -\frac{\partial \phi(x_1(0))}{\partial x_1(0)} = \frac{1}{2\gamma} x_1(0), \\ \dot{\lambda}_2(t) &= -\frac{\partial H}{\partial x_2(t)} = -\lambda_1(t), \quad \text{and} \\ \lambda_2(0) &= 0. \end{aligned} \quad (28)$$

As in the proof of Proposition 1, we prove that $c_1(t) = 0$ for $t < t_d$ in three steps. Lemma 9 shows that $\dot{\lambda}_2(0) < 0$. Lemma 10 shows that $\dot{\lambda}_2(t) > 0$ for any t at which $\lambda_2(t) < 0$. Essentially, $\lambda_2(t)$ is convex; it starts at zero and becomes negative first, then crosses zero from below. Lemma 11 shows that there exists a crossing point $t_d \in (0, T)$ such that $\lambda_2(t_d) = 0$, and this crossing point is unique. Additionally, $\dot{\lambda}_2(t_d) > 0$; thus, we have $\lambda_2(t) > 0$ for $t > t_d$. Hence, the cash allocation rule specified in Proposition 2 is optimal. \square

The switching function $\lambda_2(t)$ and the manager's payment $c_1(t)$ are depicted in Figure A.1.

LEMMA 9. $\dot{\lambda}_2(0) < 0$.

²⁰ If $\lambda_2(t) = 0$, all the derivatives of $\lambda_2(t)$ equal zero. In particular, $\dot{\lambda}_2(t) = 0$ yields $c_1(t) = b$.

PROOF. Given that $c_1(\tau) \geq 0$ for any $\tau \in [0, T]$, we have

$$\dot{\lambda}_2(0) = -\lambda_1(0) \equiv -\frac{1}{2\gamma} \int_{s=0}^T \int_{\tau=s}^T c_1(\tau) d\tau ds \leq 0.$$

We prove by contradiction that $\dot{\lambda}_2(0) < 0$. Suppose $\dot{\lambda}_2(0) = 0$. By (28), we have

$$\frac{1}{2\gamma} x_1(0) = \frac{1}{2\gamma} \int_{s=0}^T \int_{\tau=s}^T c_1(\tau) d\tau ds = 0,$$

which requires that $c_1(\tau) = 0$ for any $\tau > 0$. Thus, we have $x_1(t) = 0$ for any $t > 0$. By (28), this implies

$$\ddot{\lambda}_2(\tau) = -\dot{\lambda}_1(\tau) = s_1 b(T - \tau)^2 > 0.$$

Thus, $\lambda_2(t)$ is convex. Additionally, we have $\lambda_2(0) = 0$ and $\dot{\lambda}_2(0) = 0$. The convexity of $\lambda_2(t)$ yields $\lambda_2(\tau) > 0$ for any $\tau > 0$, which in turn requires that $c_1(\tau) = b$ for any $\tau > 0$; a contradiction! \square

LEMMA 10. $\ddot{\lambda}_2(t) > 0$ for any t at which $\lambda_2(t) < 0$.

PROOF. Given that $c_1(\tau) \leq b$ for any $\tau \in [0, T]$, we have

$$\begin{aligned} \ddot{\lambda}_2(t) &= -\dot{\lambda}_1(t) = s_1 b(T - t)^2 - 2s_1 x_1(t) \\ &= s_1 b(T - t)^2 - 2s_1 \int_{s=t}^T \int_{\tau=s}^T c_1(\tau) d\tau ds \geq 0. \end{aligned}$$

We prove by contradiction that $\ddot{\lambda}_2(t) > 0$ for any $\lambda_2(t) < 0$. Suppose $\ddot{\lambda}_2(t) = 0$ for some t at which $\lambda_2(t) < 0$; then by (28), we have $c_1(\tau) = b$ for any $\tau \in (t, T]$. However, given that $\lambda_2(t) < 0$, there exists $\delta > 0$ such that $\lambda_2(\tau) < 0$ for any $\tau \in [t, t + \delta]$. This requires that $c_1(\tau) = 0$ for $\tau \in [t, t + \delta]$; a contradiction! \square

LEMMA 11. There exists a unique $t_d \in (0, T)$ such that the switching function crosses zero at t_d ; i.e., $\lambda_2(t_d) = 0$; and remains positive afterward; i.e., $\lambda_2(t) > 0$ for $t > t_d$.

PROOF. By Lemmas 9 and 10, we have $\dot{\lambda}_2(0) < 0$, and $\ddot{\lambda}_2(t) > 0$ for any t at which $\lambda_2(t) < 0$. Thus, $\lambda_2(t)$ starts from zero, becomes negative first, and crosses zero from below. Denote $t_d = \inf\{t > 0: \lambda_2(t) \geq 0\}$. We prove by contradiction that $\dot{\lambda}_2(t_d) > 0$. Suppose not. Then $\lambda_2(t_d) \leq 0$. Given that $\dot{\lambda}_2(0) < 0$ and $\ddot{\lambda}_2(t) > 0$ for $\lambda_2(t) < 0$ (i.e., $t \in (0, t_d)$), we have $\lambda_2(t) < 0$ for any $t \in (0, t_d)$. Given that $\lambda_2(0) = 0$, we have $\lambda_2(t_d) < 0$; a contradiction!

The combination of $\lambda_2(t_d) = 0$, $\dot{\lambda}_2(t_d) > 0$, and $\ddot{\lambda}_2(t) \geq 0$ for $t \in [0, T]$ implies $\lambda_2(t) > 0$ for $t > t_d$. Thus, we have $c_1(t) = 0$ for $t < t_d$ and $c_1(t) = b$ for $t \geq t_d$. The state variable $x_1(t)$ satisfies

$$x_1(t) = \frac{1}{2} b((T - t)^2 - (t_d - t)^2) \quad \text{if } t \leq t_d. \quad (29)$$

Substituting (29) into (28), we derive the switching function $\lambda_2(t)$ for $t \leq t_d$ as

$$\lambda_2(t) = -\frac{1}{4\gamma} b(T^2 - t_d^2)t + \frac{1}{12} s_1 b(6t_d^2 t^2 - 4t_d t^3 + t^4).$$

The condition that $\lambda_2(t_d) = 0$ yields $t_d = 0$ or

$$P(t_d) \equiv t_d^3 + \frac{1}{\gamma s_1} t_d^2 - \frac{1}{\gamma s_1} T^2 = 0. \quad (30)$$

There is a unique $t_d \in (0, T)$ that solves (30) because

$$P(t_d | t_d = 0) = -\frac{1}{\gamma s_1} T^2 < 0,$$

$$P(t_d | t_d = T) = T^3 > 0, \quad \text{and}$$

$$P'(t_d) = 3t_d^2 + \frac{2}{\gamma s_1} t_d > 0 \quad \text{for } t_d > 0. \quad \square$$

Efficiency Loss in the Three-Sided Moral Hazard Model

Applying the first-order conditions directly to the expected social surplus given in (2) yields the first-best effort choices as

$$e_0^{fb} = \frac{1}{4\gamma} bT^2 \quad \text{and} \quad e_i^{fb}(t) = \frac{m_i}{4k_i} b(T - t)^2 \quad \text{for } i = 1, 2.$$

Substituting these first-best effort levels into (2), we obtain the first-best social surplus as

$$\Pi^{fb} = \left(bT - \frac{1}{2} \phi bT^2 - \frac{1}{6} (m_1 + m_2) bT^3 + \frac{1}{16\gamma} b^2 T^4 \right) + \frac{1}{10} s_1 b^2 T^5.$$

Substituting the second-best effort given in (7) into (2), we obtain the second-best social surplus as

$$\begin{aligned} \Pi^{sb} &= \left(bT - \frac{1}{2} \phi bT^2 - \frac{1}{6} (m_1 + m_2) bT^3 + \frac{1}{16\gamma} b^2 T^4 \right) \\ &\quad - \frac{1}{16\gamma} b^2 (T^2 - t_d^2)^2 \\ &\quad + s_1 b^2 \left(\frac{3}{40} T^5 - \frac{1}{12} T^2 t_d^3 + \frac{1}{24} T t_d^4 - \frac{1}{30} t_d^5 \right). \end{aligned}$$

The welfare loss is thus

$$\begin{aligned} \Pi^{fb} - \Pi^{sb} &= \frac{1}{16\gamma} b^2 (T^2 - t_d^2)^2 \\ &\quad + s_1 b^2 \left(\frac{1}{40} T^5 + \frac{1}{12} T^2 t_d^3 - \frac{1}{24} T t_d^4 + \frac{1}{30} t_d^5 \right), \end{aligned}$$

which is increasing with the severity of the agents' moral hazard problems, $(1/\gamma, s_1)$. \square

Efficiency Loss in the Sequential Compensation Model

Applying the first-order conditions directly to the expected social surplus (9), we have the first-best effort choices as

$$e_0^{fb} = \frac{1}{4\gamma} bT^2 \quad \text{and} \quad e_1^{fb}(t) = \frac{m_1}{4k_1} b(T - t)^2.$$

Substituting these first-best effort levels into (9), we obtain the first-best social surplus as

$$\Pi^{fb} = bT - \frac{1}{2} \phi bT^2 - \frac{1}{6} m_1 bT^3 + \frac{1}{16\gamma} b^2 T^4 + \frac{1}{20} s_1 b^2 T^5.$$

Substituting the second-best effort given in (13) into (9), we obtain the second-best social surplus as

$$\begin{aligned} \Pi^{sb} &= bT - \frac{1}{2} \phi bT^2 - \frac{1}{6} m_1 bT^3 + \frac{1}{8\gamma} b^2 t_d^2 T^2 \\ &\quad - \frac{1}{16\gamma} b^2 t_d^4 + \frac{1}{20} s_1 b^2 (T^5 - t_d^5). \end{aligned}$$

The welfare loss is

$$\Pi^{fb} - \Pi^{sb} = \frac{1}{16\gamma} b^2 (T^2 - t_d^2)^2 + \frac{1}{20} s_1 b^2 t_d^5 > 0,$$

which is increasing as the severity of the moral hazard problems increases, $(1/\gamma, s_1)$. \square

Efficiency Loss in the Proportional Sharing Model

Applying the first-order conditions directly to the expected social surplus (15), we have the first-best effort choices as

$$e_1^{fb}(t) = \frac{m_1}{4k_1} b(T-t)^2 \quad \text{and} \quad e_2^{fb}(t) = \frac{m_2}{4k_2} b(T-t)^2.$$

Substituting these first-best effort levels into (15), we obtain the first-best social surplus as

$$\Pi^{fb} = bT - \frac{1}{6}(m_1 + m_2)bT^3 + \frac{1}{20}(s_1 + s_2)b^2T^5.$$

Substituting the second-best effort given in (18) into (15), we obtain the second-best social surplus as

$$\Pi^{sb} = bT - \frac{1}{6}(m_1 + m_2)bT^3 + \frac{1}{20} \frac{s_1^2 + s_1s_2 + s_2^2}{s_1 + s_2} b^2T^5.$$

Thus, the welfare loss is

$$\Pi^{fb} - \Pi^{sb} = \frac{1}{20} \frac{s_1s_2}{s_1 + s_2} b^2T^5,$$

which is increasing as the severity of the moral hazard problems, (s_1, s_2) , increases. \square

Proportional Sharing Model with Three Agents

In a case with three agents all exerting effort over time, cash allocations to the agents (for $t \in [0, T]$) are

$$c_1^*(t) = \frac{s_1s_2 + s_1s_3 - s_2s_3}{s_1s_2 + s_1s_3 + s_2s_3} b, \quad c_2^*(t) = \frac{s_1s_2 + s_2s_3 - s_1s_3}{s_1s_2 + s_1s_3 + s_2s_3} b,$$

$$\text{and} \quad c_3^*(t) = \frac{s_1s_3 + s_2s_3 - s_1s_2}{s_1s_2 + s_1s_3 + s_2s_3} b,$$

where $c_i^*(t)$ (for $i = 1, 2, 3$) is the amount of proceeds allocated to agent i at time t , which increases with the severity of the agent's moral hazard, s_i .

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